

CRYSTALLOGRAPHIC FLAT ORIGAMIS

By

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Origami is an art of paper-foldings. From a sheet of paper, they produce a bird, an animal, a plane, a ship and etc. Origami is transmitted (especially in Japan) as a popular amusement, a traditional art (eg. [5]) and as a method of mathematical education (eg. [2], [3], [7]). In recent years there are several mathematical studies on origami (eg. [4], [6]). In order to develop the mathematical theory further, the authors think that it is necessary to give a firm background to origami. To begin with, we give in this paper a rigorous (and at the same time practical) definition of *flat origamis* and their foldings, where a flat origami is an origami of which finished work is flat. Since origamis which are invariant under some group actions are of special interest, we study their properties. We also give some examples of flat origamis with high symmetry which are called *crystallographic flat origamis*.

§1. Flat Origamis

Let X be the real euclidean plane and $\text{Isom}(X)$ be the group of isometric automorphisms. For a segment (a piece of a line) l in X , we denote by $R(l) \in \text{Isom}(X)$ the reflection whose axis is the line including l .

Definition: A locally finite cell decomposition K of X is called a *flat origami* if for an arbitrary closed curve γ in X such that γ does not pass through any vertex of K and intersects edges (1-cells) $l_1, \dots, l_r$ of K transversally in this order, the following condition holds:

$$R(l_1) \cdots R(l_r) = \text{identity}.$$

Remark: A flat origami K represents the “development chart” of an origami though its valley creases and its mountain creases are not distinguished.

For two faces (2-cells) σ and σ' of a flat origami K , let γ be a curve starting from σ and ending at σ' such that γ does not pass through any vertex of K and intersects edges $l_1, \dots, l_r$ of K transversally in this order. We define the transformation $[\sigma', \sigma] \in \text{Isom}(X)$ as follows:

$$[\sigma, \sigma'] = R(l_1) \cdots R(l_r).$$

Notice that the transformation is independent of the choice of the curve γ , in view of the definition of flat origami. Notice also that for three faces $\sigma, \sigma', \sigma''$ of K , we have

$$[\sigma, \sigma'] [\sigma, \sigma''] = [\sigma, \sigma''].$$

Definition: Let σ be a face of a flat origami K . The folding map $[\sigma]$ of K toward σ is a piecewise isometric map defined as follows:

$$[\sigma](x) = [\sigma, \sigma'](x) \quad \text{for } x \in \sigma' \in K.$$

Notice that we have $[\sigma'] = [\sigma', \sigma] [\sigma]$ and that the map $[\sigma]$ is not injective unless K is empty.

§2. Symmetries of origamis

Let Γ be a subgroup of $\text{Isom}(X)$ which leaves K invariant. We call K a Γ -invariant flat origami. We study the symmetry of a flat origami after it is folded.

Definition: Let K be a Γ -invariant flat origami, and let σ be a face of K . We define a map $\varphi_\sigma: \Gamma \rightarrow \text{Isom}(X)$ by

$$\varphi_\sigma(g) = [\sigma, g\sigma]g \quad \text{for } g \in \Gamma.$$

PROPOSITION: The map φ_σ is a homomorphism.

Proof: Notice first that we have $[g\sigma, g\sigma'] = g[\sigma, \sigma']g^{-1}$ for $\sigma, \sigma' \in K, g \in \Gamma$. Then for $g, h \in \Gamma$, we have $\varphi_\sigma(gh) = [\sigma, gh\sigma]gh = [\sigma, g\sigma][g\sigma, gh\sigma]gh = [\sigma, g\sigma]g[\sigma, h\sigma]g^{-1}gh = [\sigma, g\sigma]g[\sigma, h\sigma]h = \varphi_\sigma(g)\varphi_\sigma(h)$. $\square$

PROPOSITION: For two faces σ, σ' of K , homomorphisms φ_σ and $\varphi_{\sigma'}$ are conjugate.

Proof: $\varphi_{\sigma'}(g) = [\sigma', g\sigma']g = [\sigma', \sigma][\sigma, g\sigma][g\sigma, g\sigma']g = [\sigma', \sigma][\sigma, g\sigma]g[\sigma, \sigma'] = [\sigma', \sigma]\varphi_\sigma(g)[\sigma', \sigma]^{-1}$. $\square$

The following proposition shows that the image group $\varphi_\sigma(\Gamma)$ describes the symmetry of the finished work of a Γ -invariant flat origami.

PROPOSITION: $\varphi_\sigma(g)[\sigma] = [\sigma]g$.

Proof: For $x \in \sigma'$, we have $\varphi_\sigma(g)[\sigma]x = [\sigma, g\sigma]g[\sigma, \sigma']x = [\sigma, g\sigma][g\sigma, g\sigma']gx = [\sigma, g\sigma']gx$. $\square$

§ 3. Crystallographic flat origamis

A discrete subgroup Γ of $\text{Isom}(X)$ is called a (2-dimensional) *crystallographic group* if X modulo Γ has finite volume.

Definition: A flat origami which is invariant under a crystallographic group is called a *crystallographic flat origami*.

The Bieberbach theorem for crystallographic groups and the fact that crystallographic groups have no finite normal subgroups lead to the following

PROPOSITION: Let K be a flat origami invariant under a crystallographic group Γ . If the image $\varphi_\sigma(\Gamma)$ is a crystallographic group then φ_σ is an isomorphism into $\text{Isom}(X)$.

§ 4. Concluding remarks.

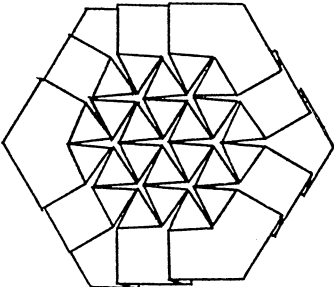
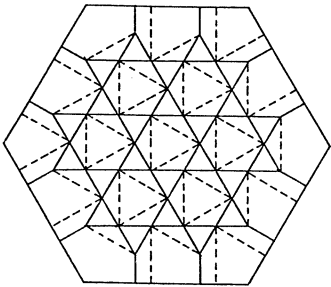
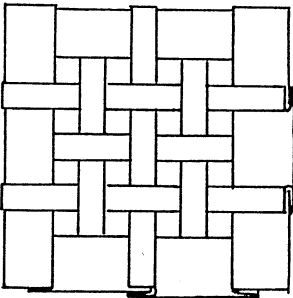
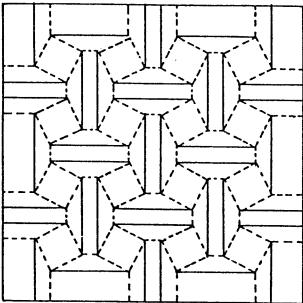
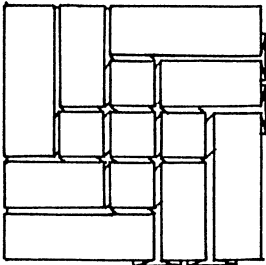
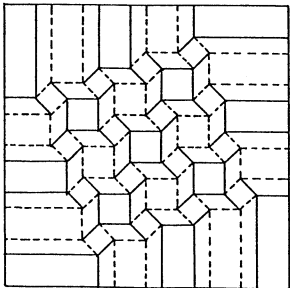
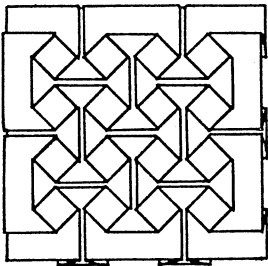
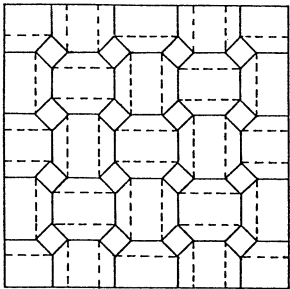
It is known that for every crystallographic group Γ there is a flat origami such that Γ is the maximal subgroup of $\text{Isom}(X)$ which leaves K invariant. For some special Γ , there is an algorithm to construct all Γ -invariant flat origamis. These will be published elsewhere.

Examples of crystallographic flat origamis

We give four examples of crystallographic flat origamis together with their finished works. Dashed lines denote valley creases while lines denote mountain creases.

K

finished work of K



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