

On High Dimensional Flat Origamis

By

Toshikazu Kwasaki

§ 0. Notation.

Let X be the real euclidian line R , plane $R^2, \dots$, n -dimensional space R^n , unit circle S^1 , unit 2-sphere $S^2, \dots$, or unit n -sphere S^n . Let $\text{Isom}(X)$ be the group of isometric automorphisms. For a segment Q (if $X=R$), a face f (if $X=R^2$), $\dots$, a hyperplane σ (if $X=R^n$), we denote by $R(Q) \in \text{Isom}(R^2)$, $R(f) \in \text{Isom}(R^2)$, $\dots$, $R(\sigma) \in \text{Isom}(R^n)$, the reflection whose axis is the line including Q , whose mirror is the plane including $f, \dots$, whose mirror is the hyperplane including σ (Fig.0).

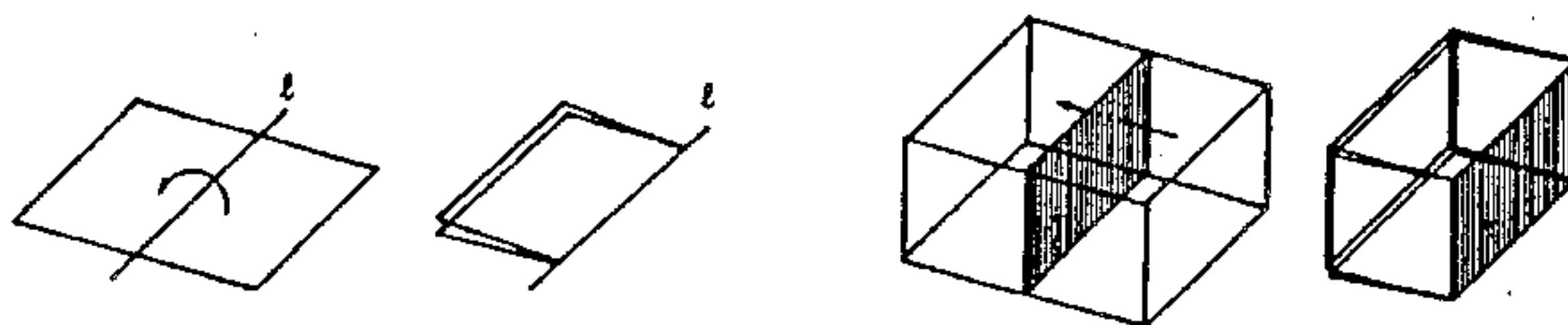


Fig.0 (reflection)

§ 1. Flat origamis of R^2, S^2 and R^3

1.1. Flat origamis of R^2

Definition 1.1 (flat origamis of R^2): Let X be R^2 . A locally finite cell decomposition K of X is called a flat origami if for an arbitrary closed curve γ in X such that γ does not pass through any vertex of K and intersects 1-cells (edges) $Q_1, \dots, Q_r$ of K transversally in this order, the following condition (We call this condition by "flat condition") holds:

$$R(Q_1) \cdots R(Q_r) = \text{identity}.$$

Remark: A flat origami K represents the "development chart" of an origami though its valley creases and its mountain creases are not distinguished.

1.2. Flat origamis of R^3

Definition 1.2 (flat origamis of R^3): A locally finite cell decomposition K of X is called a flat origami if for an arbitrary closed curve γ in X such that γ does not pass through any 0 or 1-cell of K and intersects 2-cells $\sigma_1, \dots, \sigma_r$ of K transversally in this order, the flat condition holds:

$$R(\sigma_1) \cdots R(\sigma_r) = \text{identity}.$$

1.3. Example of flat origamis of R^3

For a flat origami K of R^3 and a integer $m=0, 1, 2, 3$, we denote by K_m the set of m -cells of K .

Example 1.3 (Fig.1.1 (1)) K_2 consists of 12 quarter planes of xy , yz and zx -plane. K_1 consists of 6 half lines concluded in x -axis, y -axis and z -axis. K_0 consists of only the origin.

Let γ be the closed curve such that γ passes through faces $f_1, \dots, f_6$ transversally as shown in Fig. 1.1 (2). It follows :

$$\begin{aligned} R(f_1) = R(f_3) &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ R(f_2) = R(f_5) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ R(f_4) = R(f_6) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

We can see the product of the above reflections:

$$R(f_1) \cdots R(f_6) = \text{identity}.$$

Example 1.4 (Fig.1.2) We let join two flat origamis like Example 1.3. Then we get another flat origami.

The above two flat origamis are trivial. Next example is non-trivial flat origami.

Example 1.5 (Fig.1.3)

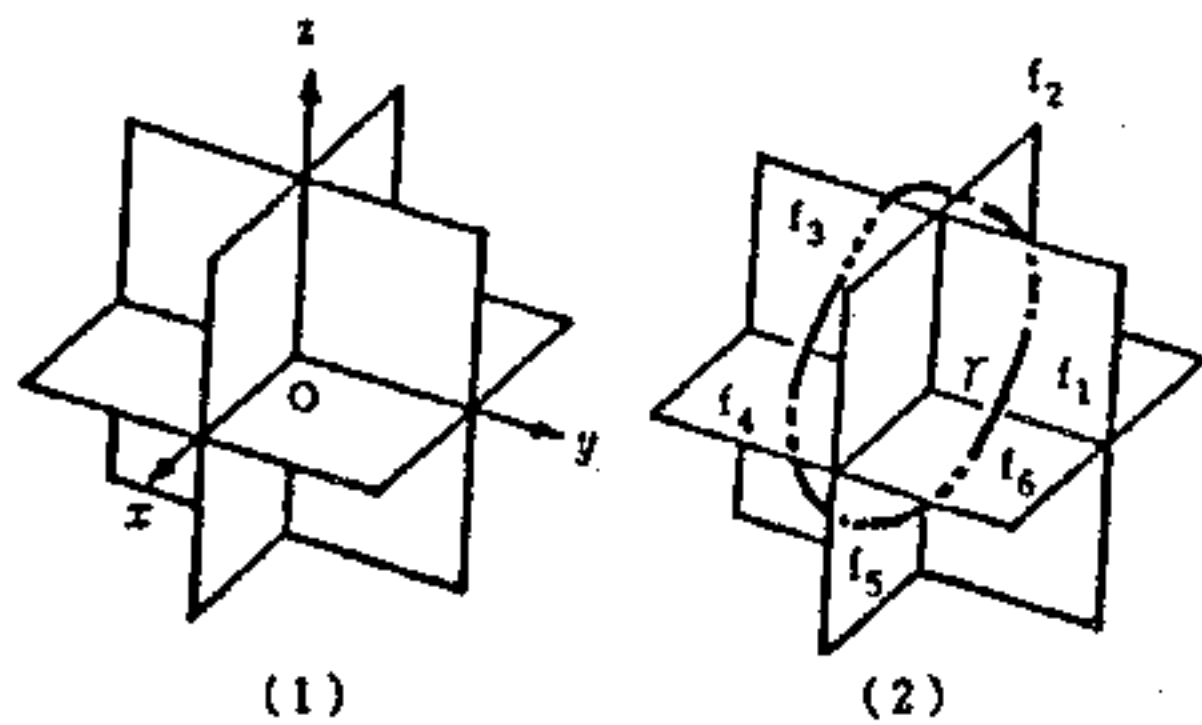


Fig.1.1 (a flat origami of R^3)

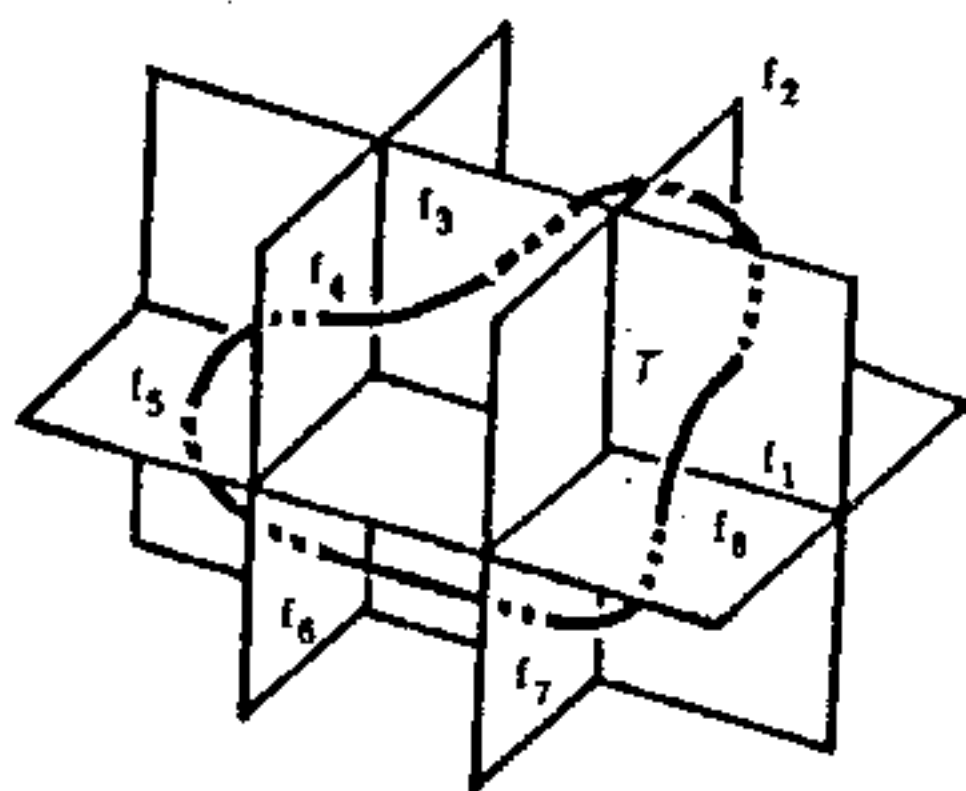
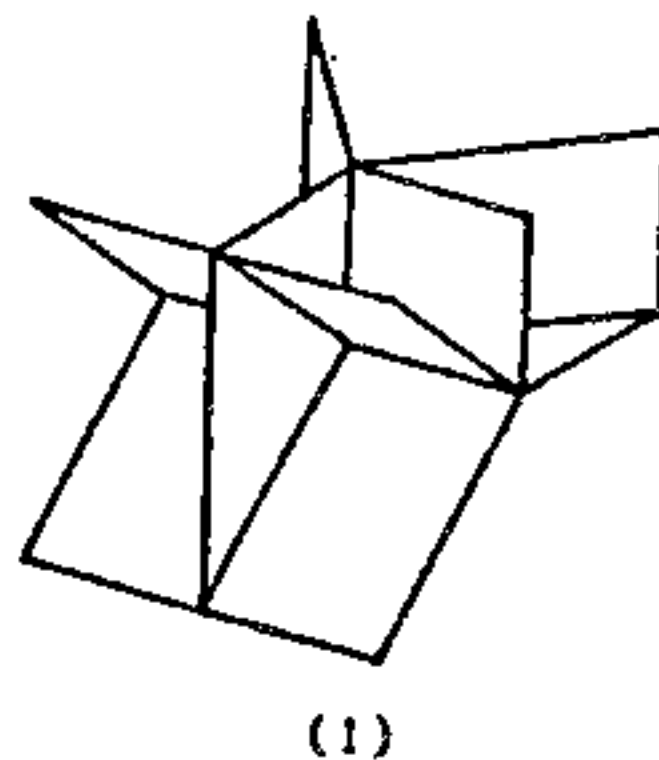
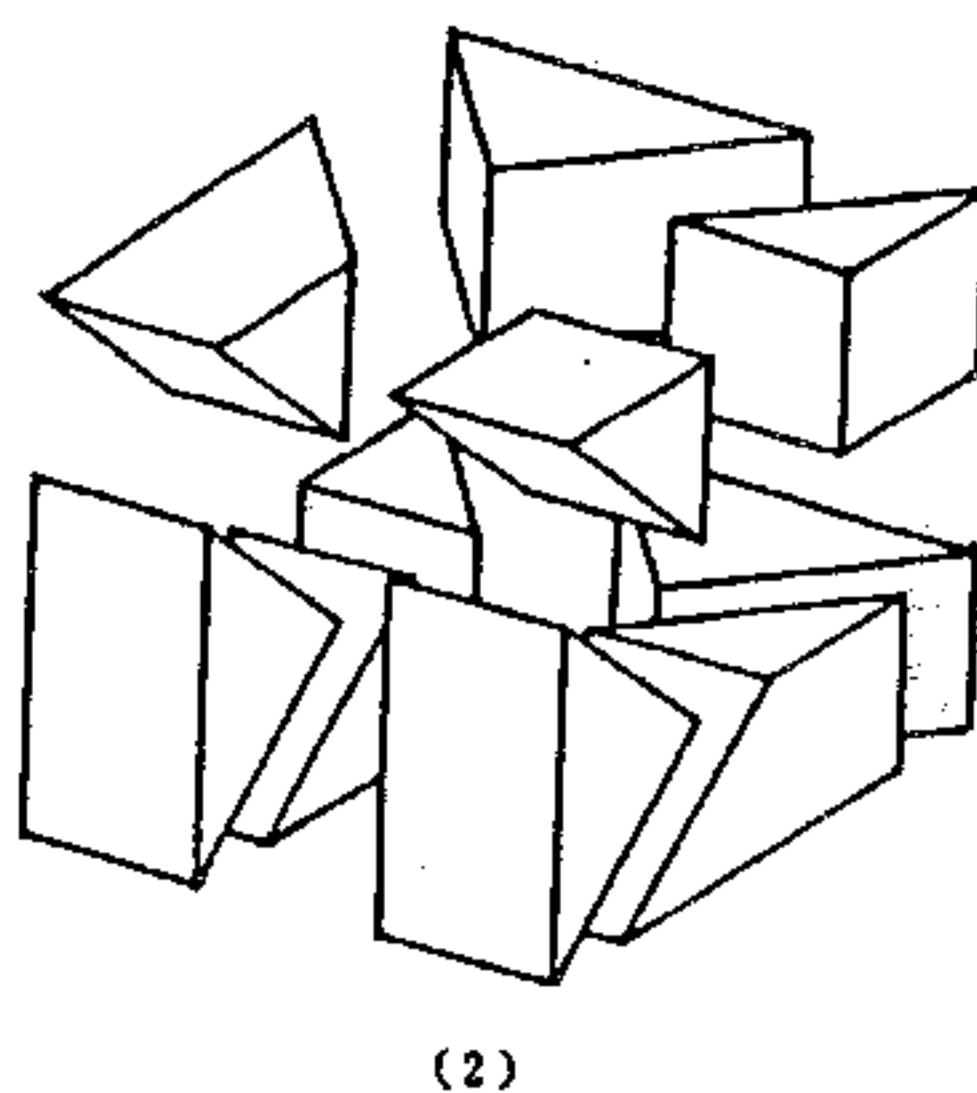


Fig.1.2 (a flat origami of R^3)



(2)

Fig.1.3 (a flat origami of R^3)

1.4. Localization of flat origamis of R^2

Let X be R^2 , K be a locally finite cell decomposition of X , p be a 0-cell of K .

Definition 1.6 (localization at p) Let $K(p)$ be the cell decomposition whose cells are the cones with vertex p generated by the cells including p . We call it by localization of K at p .

Remark: If cells do not include p , then they vanish by localization at p . If a cell include p , then a 1-cell Q (a segment) $\rightarrow$ a half line including Q , a 2-cell f (a face) $\rightarrow$ the conical hull of the polygon f with vertex p .

Proposition 1.7 For a cell-decomposition of K ,

K is a flat origami R^2

$\Leftrightarrow$ For any 0-cell p , $K(p)$ is a flat origami.

1.5. Localization of flat origamis of R^3

Let X be R^3 , K be a locally finite cell decomposition of X , p be a 0-cell of K and Q be a 1-cell (a segment, a half line or a line).

Definition 1.8 (localization at p) Let $K(p)$ be the cell decomposition whose cells are the cones with vertex p generated by the cells (including p) of K . We call it by localization of K at p (Fig.1.4, 1.5.).

Remark: If cells do not include p , then they vanish by localization at p . If a cell include p , a 1-cell $Q \rightarrow$ a half line including Q , 2-cell f (a face) $\rightarrow$ the conical hull of the polygon f with vertex p , 3-cell σ (a polyhedron) $\rightarrow$ the conical hull of σ with vertex p .

Definition 1.9 (localization at Q) Let $K(Q)$ be the cell decomposition whose cells are the cones with Q generated by the cells including Q . We call it by localization of K at Q (Fig.1.4, 1.6.).

Remark: If cells do not include Q , then they vanish by localization at Q . If a cell include Q , $Q \rightarrow$ the line including Q , 2-cell $f \rightarrow$ the half plane with line including both Q and f , 3-cell $\sigma \rightarrow$ the conical hull of σ with line including Q .

Proposition 1.10 For a cell-decomposition K ,
 K is a flat origami of R^3

- $\Leftrightarrow$ For any 0-cell p , $K(p)$ is a flat origami.
- $\Leftrightarrow$ For any 1-cell Q , $K(Q)$ is a flat origami.

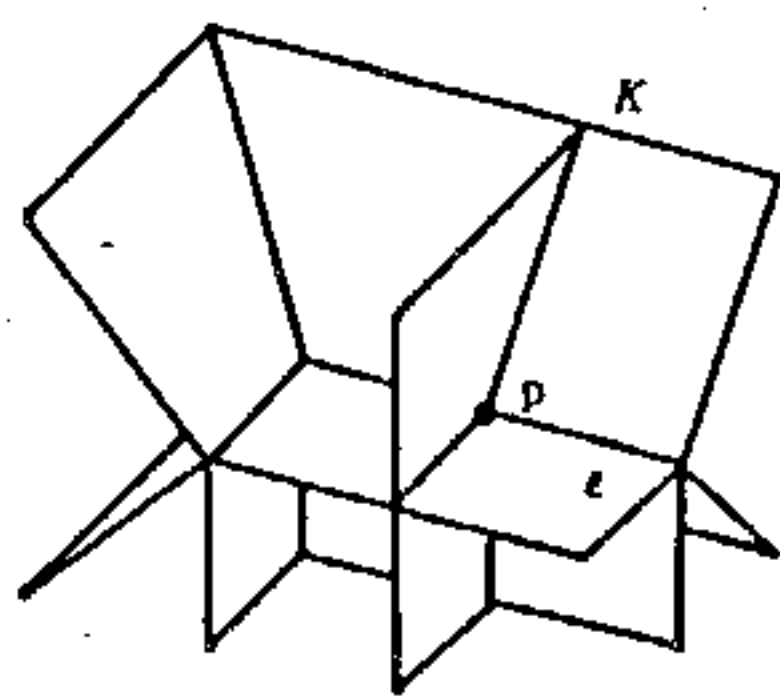


Fig.1.4

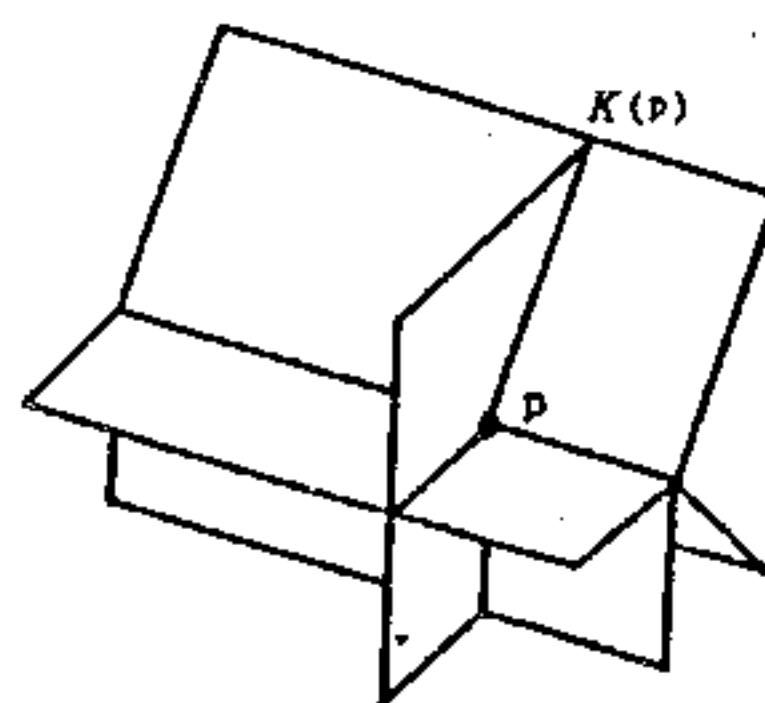


Fig.1.5

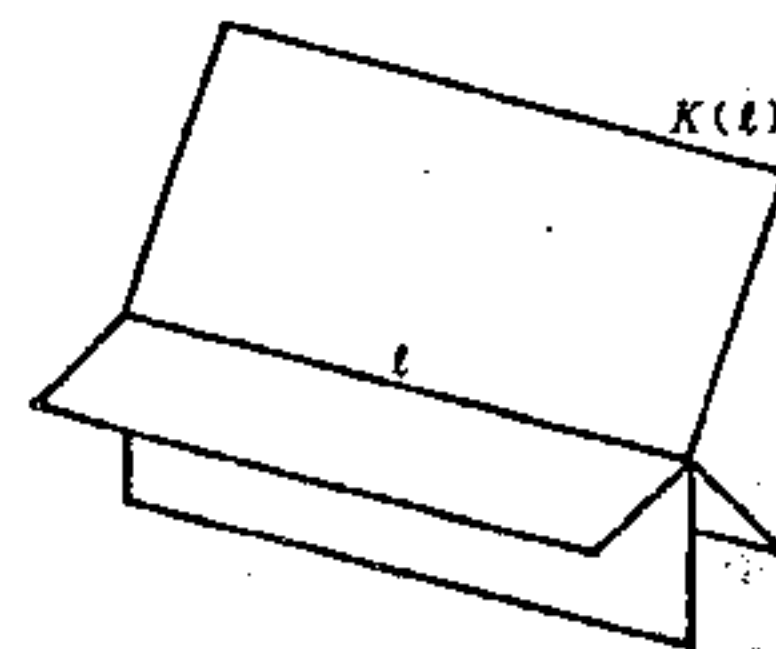


Fig.1.6

(localization at p) (localization at Q)

1.6. Flat origamis of 2-sphere S^2

Let $\mathcal{K}$ be a locally finite cell decomposition (whose 1-cell is a piece of a great circle: an arc) of S^2 . For an arc m in S^2 , we denote by $R(m) \in \text{Isom}(S^2 \cap R^3)$ the restriction of the reflection whose mirror is the plane including m .

Definition 1.11 (flat origamis of 2-sphere S^2): A locally finite cell decomposition $\mathcal{K}$ of S^2 is called a flat origami if for an arbitrary closed curve γ in S^2 such that γ does not pass through any vertex of $\mathcal{K}$ and intersects 1-cells $m_1, \dots, m_r$ of $\mathcal{K}$ transversally in this order, the flat condition holds:

$$R(m_1) \cdots R(m_r) = \text{identity}.$$

Remark: Let q be a vertex of a flat origami $\mathcal{K}$ of S^2 and arcs $m_1, \dots, m_{2n}$ start from q in this order. It follows that

$$\alpha_1 + \alpha_3 + \alpha_5 + \cdots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \alpha_6 + \cdots + \alpha_{2n} = 2\pi.$$

where α_1 is the angle between m_1 and m_{1+1} , α_{2n} is the angle between m_{2n} and m_1 .

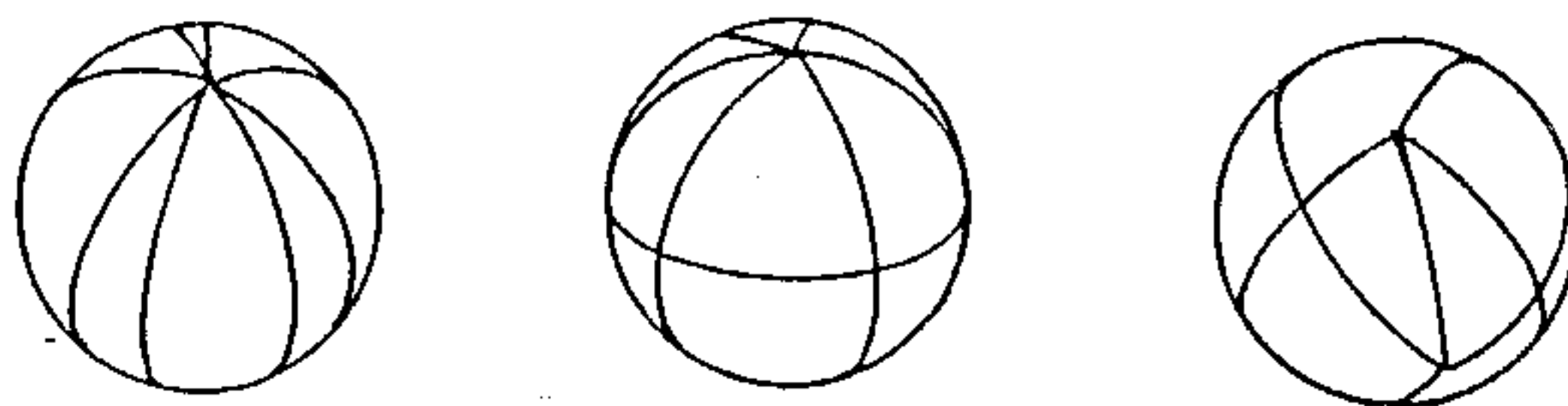


Fig.1.7 (flat origamis of S^2)

1.7. Localization of flat origamis of S^2

Let $\mathcal{K}$ be a locally finite cell decomposition of S^2 , q be a 0-cell of $\mathcal{K}$, q' be the antipodal point of q .

Definition 1.12 (localization at q) Let $\mathcal{K}(q)$ be the cell decomposition whose cells are the cone (in S^2) with vertex q generated by the cells including q . We call it by localization of $\mathcal{K}$ at q (Fig.1.8).

Remark: If cells do not include q , then they vanish by localization. If

a cell include q , a 1-cell $m \rightarrow$ a half of the great circle including m ,
2-cell (spherical polygon) $\sigma \rightarrow$ intersection S^2 and the cone (in R^3)
with the segment qq' generated by σ . Moreover, $(\mathbb{k}(q))_0 = \{q, q'\}$.

Proposition 1.13 For a cell decomposition $\mathbb{k}$ of S^2 ,
 $\mathbb{k}$ is a flat origami
 $\Leftrightarrow$ For any 0-cell q , $\mathbb{k}(q)$ is a flat origami.

1.8. Cutting by R or S^2 .

Let S^2 be the sphere with the center p .

Proposition 1.14 If K is a flat origami of R^3 , then intersection
 $K \cap S^2$ is a flat origami of S^2 .

Definition 1.15 (cutting by S^2) Intersection $K \cap S^2$ is called by the
cutting (by S^2) of K (Fig.1.9).

Definition 1.16 (cutting by R) For a locally finite cell decomposition
 K of R^3 , a 1-cell Q and a plane Π which is perpendicular to Q , we call
intersection $K(Q) \cap \Pi$ by cutting (by R) of K (Fig.1.10).

Proposition 1.17 K is a flat origami $\Leftrightarrow$ intersection $K(Q) \cap \Pi$ is a
flat origami of R^2 for any 1-cell Q .

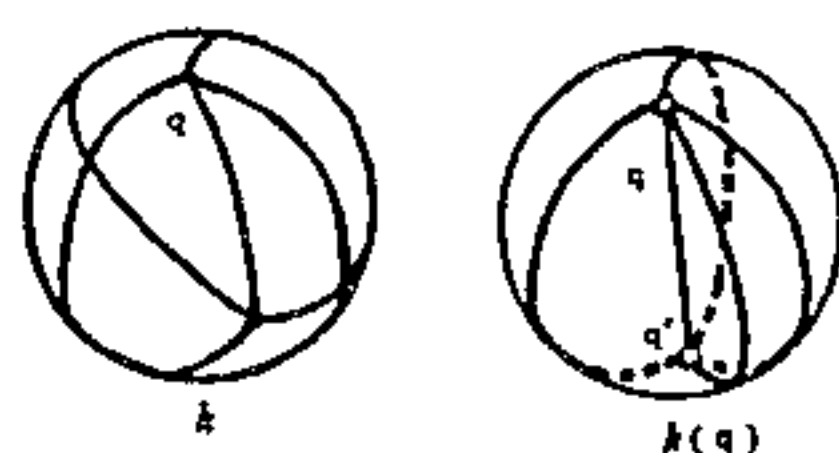


Fig.1.8 (localization at q)

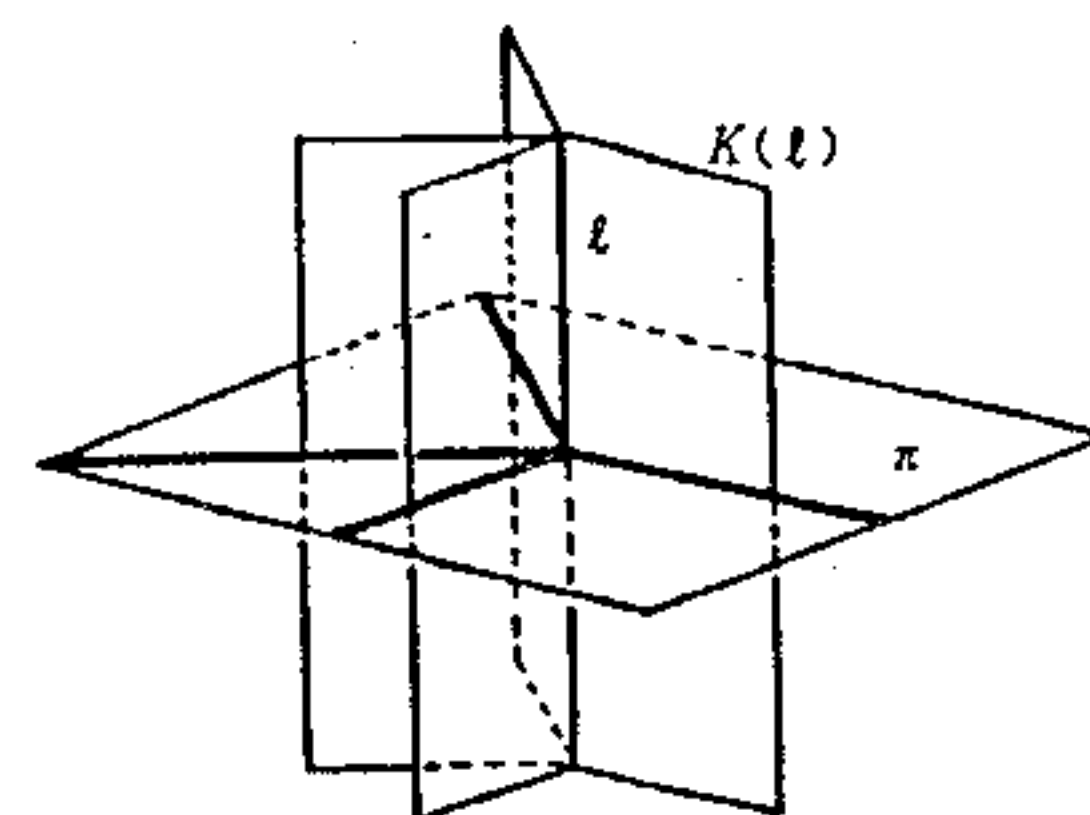


Fig.1.10 (cutting by R)

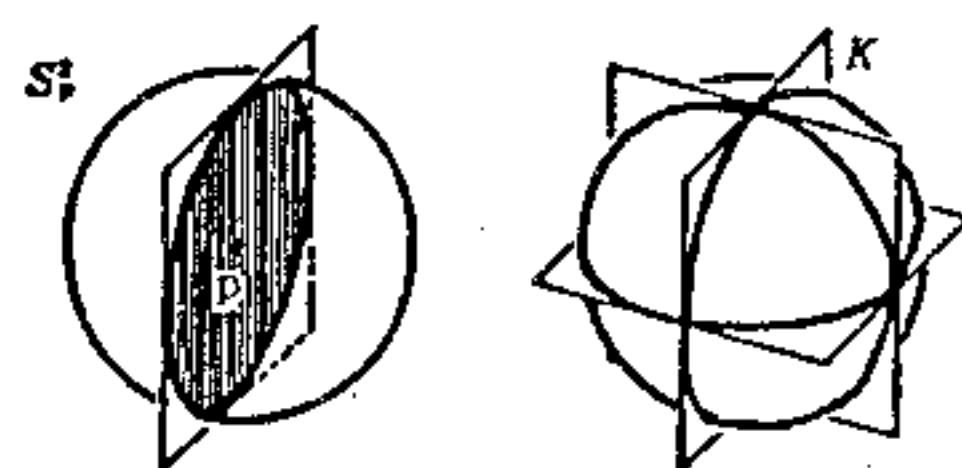


Fig.1.9 (cutting by S^2)

1.9. Projection from S^2 to R .

Let $k(q)$ be localization of a flat origami k of S^2 , π be the tangent plane at q , ρ be the projection from the center p of sphere S^2 to π (Fig.1.11).

Proposition 1.18 $\rho(k(q))$ is a flat origami of R^2 .

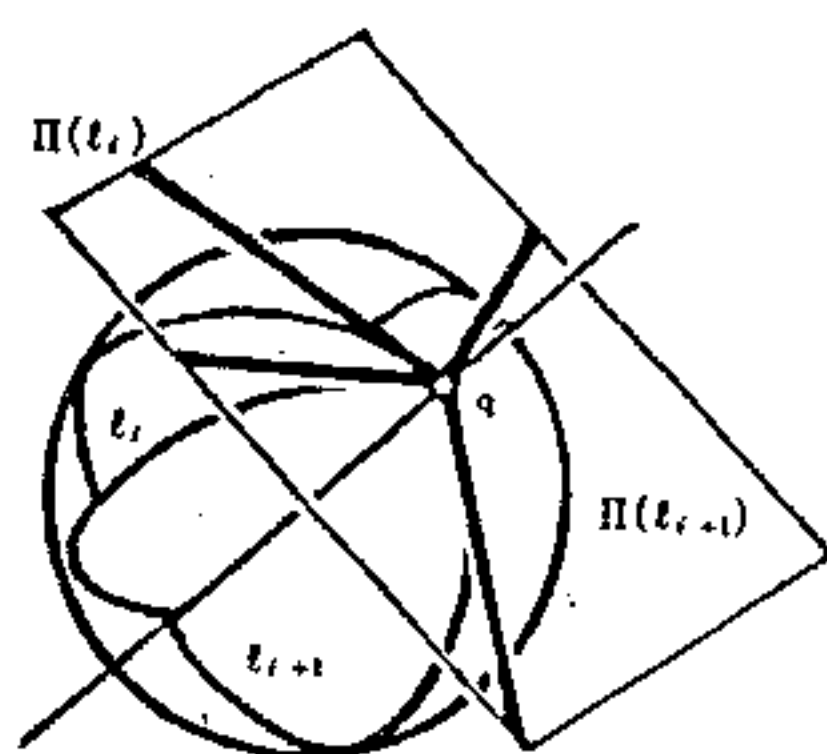


Fig.1.11 (projection)

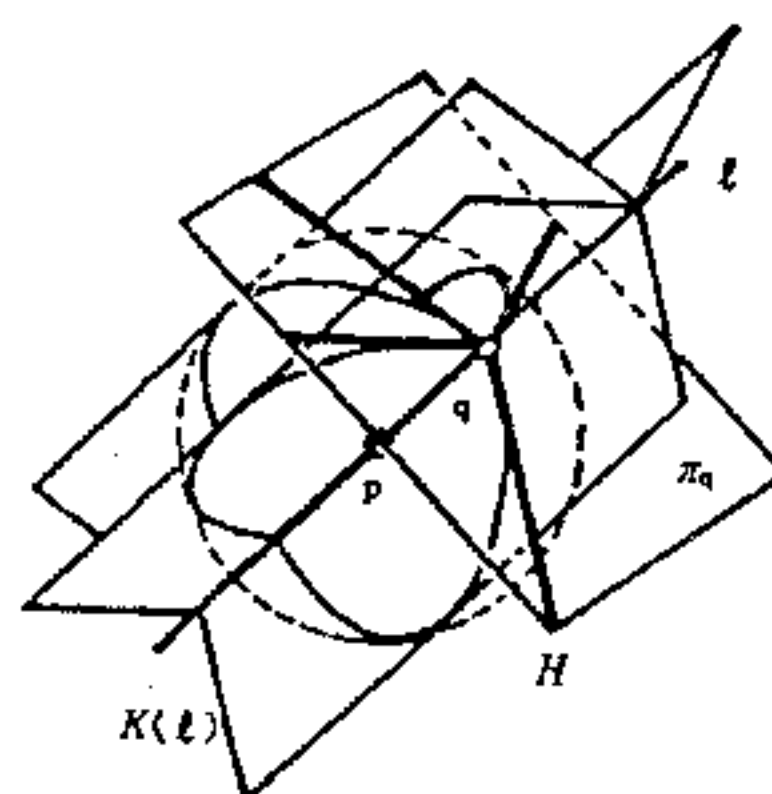
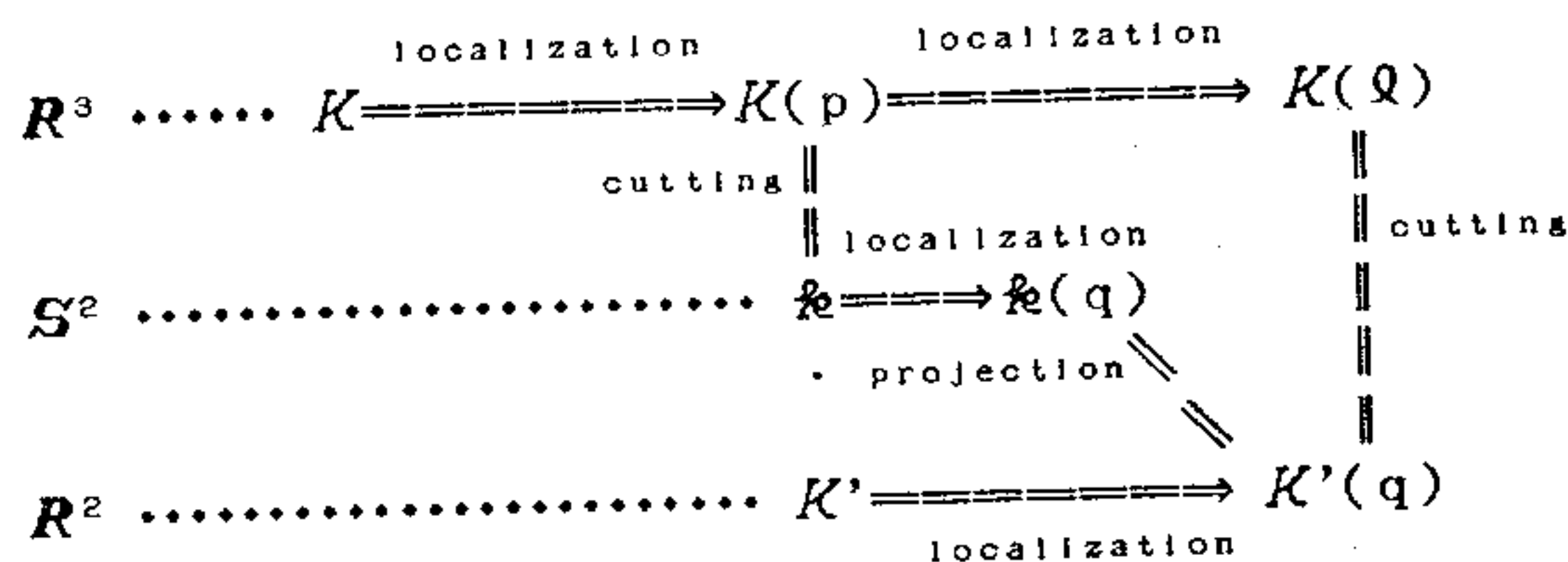


Fig.1.12

1.10. The relation between flat origamis of R^2 , S^2 and R^3 .

We can express the relation between flat origamis of R^2 , S^2 and R^3 as folloing:



§ 2. High dimensional flat origamis

2.1. Flat origamis of $X = \mathbb{R}^n$ ($n = 2, 3, 4, \dots$)

Definition 2.1 (flat origami of $\mathbb{R}^n$): A locally finite cell decomposition K of X is called a flat origami if for an arbitrary closed curve γ in X such that γ does not pass through any m ($m = 1, 2, \dots, n-2$) dimensional cell of K and intersects hyperplanes ($n-1$ cells) $\sigma_1, \dots, \sigma_r$ of K transversally in this order, the flat condition holds:

$$R(\sigma_1) \cdots R(\sigma_r) = \text{identity}.$$

For two n -cells σ, σ' of a flat origami K , let γ be a curve starting from σ and ending at σ' such that γ does not pass through any m ($m = 1, 2, \dots, n-2$) dimensional cell of K and intersects ($n-1$)-cells $\sigma_1, \dots, \sigma_r$ of K transversally in this order. We define the transformation $[\sigma, \sigma'] \in \text{Isom}(X)$ as follows:

$$[\sigma, \sigma'] = R(\sigma_1) \cdots R(\sigma_r).$$

Notice that the transformation is independent of the choice of the curve γ , in view of the definition of flat origami. Notice also that for three n -cells $\sigma, \sigma', \sigma''$ of K , we have

$$[\sigma, \sigma''] = [\sigma, \sigma'] [\sigma', \sigma''].$$

Definition 2.2 (folding map): Let σ be a n -cell of a flat origami K . The folding map $[\sigma]$ of K toward σ is a piecewise isometric map defined as follows:

$$[\sigma](x) = [\sigma, \sigma'](x) \quad \text{for } x \in \sigma' \in K.$$

Notice that we have $[\sigma'] = [\sigma', \sigma][\sigma]$ and that the map $[\sigma]$ is not injective unless K is empty.

2.2. Localization of flat origamis of $\mathbb{R}^n$

Let X be $\mathbb{R}^n$, K be a locally finite cell decomposition of X , p be a 0-cell of K .

Definition 2.3 (localization at p): Let $K(p)$ be the cell decomposition

tion whose cells are the cones with vertex p generated by the cells including p . We call it by localization of K at p .

Remark: Action of localization is similar to that described for R^3 .

Proposition 2.4 For a cell decomposition K ,

K is a flat origami of R^n

$\Leftrightarrow$ For any 0-cell p , $K(p)$ is a flat origami.

2.3. Flat origamis of n -sphere S^n

We can define flat origamis of n -sphere S^n as following:

Definition 2.5 (Flat origamis of S^n) A cell decomposition $\mathbb{k}$ of S^n is called by a flat origami, if there exists a flat origami K of R^{n+1} such that $\mathbb{k}$ is a intersection of a localization $K(p)$ and S^n with center p .

We call the above flat origami $\mathbb{k}$ by cutting of $K(p)$ (by S^n).

2.4. Localization of a flat origami of n -sphere S^n .

Let $\mathbb{k}$ be a flat origami of S^n , q is a 0-cell of $\mathbb{k}$.

Definition 2.6 (localization at q) Let $\mathbb{k}(q)$ be the cell decomposition whose cells are the cones (in S^n) with vertex p generated by the cells including q . We call it by localization of $\mathbb{k}$ at q .

Remark: Action of localization is similar that described for S^2 .

Proposition 2.7 For a cell decomposition $\mathbb{k}$ of S^n ,

$\mathbb{k}$ is a flat origami

$\Leftrightarrow$ For any 0-cell q , $\mathbb{k}(q)$ is a flat origami.

2.5. Projection from S^n to R^n .

Let $\mathbb{k}(q)$ be localization of a flat origami $\mathbb{k}$ of S^n , π be the tangent hyperplane at q , ρ be the projection from the center p of S^n to π .

Proposition 2.8 $\rho(\mathbb{k}(q))$ is a flat origami of R^n .

§ 3. Flat origamis of R , S^1 and R^2

3.1. Flat origamis of R and S^1

We can define 1 dimensional flat origamis as following:

Definition 3.1 (flat origamis of R): A locally finite cell decomposition of R is called by a flat origami of R (Fig.3.1).

Remark: If K_0 is countable, then K is a flat origami.

Definition 3.2 (flat origamis of S^1): A locally finite cell decomposition $\mathcal{K}$ of S^1 is called a flat origami if $\mathcal{K}_1 = \{ q_1, q_2, \dots, q_m \}$, m is even ($= 2n$) and it is satisfied that

$$\alpha_1 + \alpha_3 + \alpha_5 + \dots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \alpha_6 + \dots + \alpha_{2n} = 2\pi,$$

where α_i is the length of the arc $q_i q_{i+1}$, α_{2n} is the length of the arc $q_{2n} q_1$ (Fig.3.3).

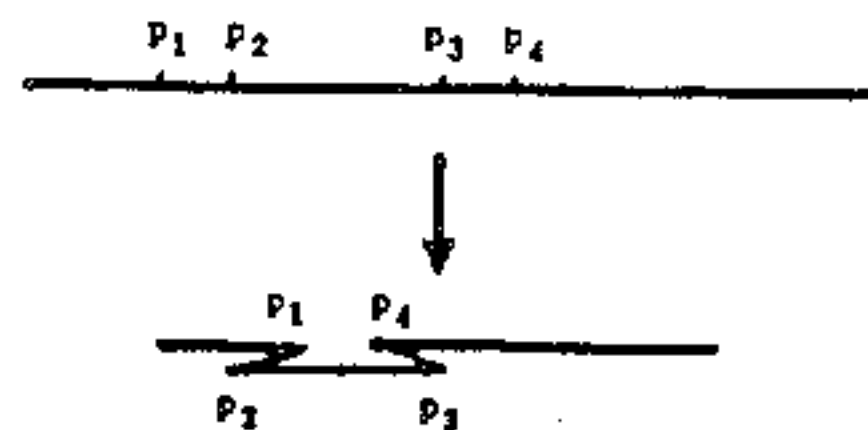


Fig.3.1 (a flat origami of R)

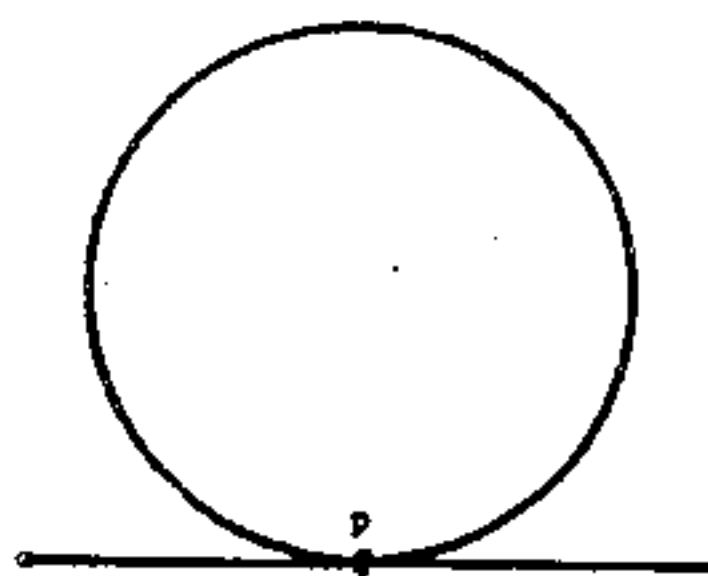
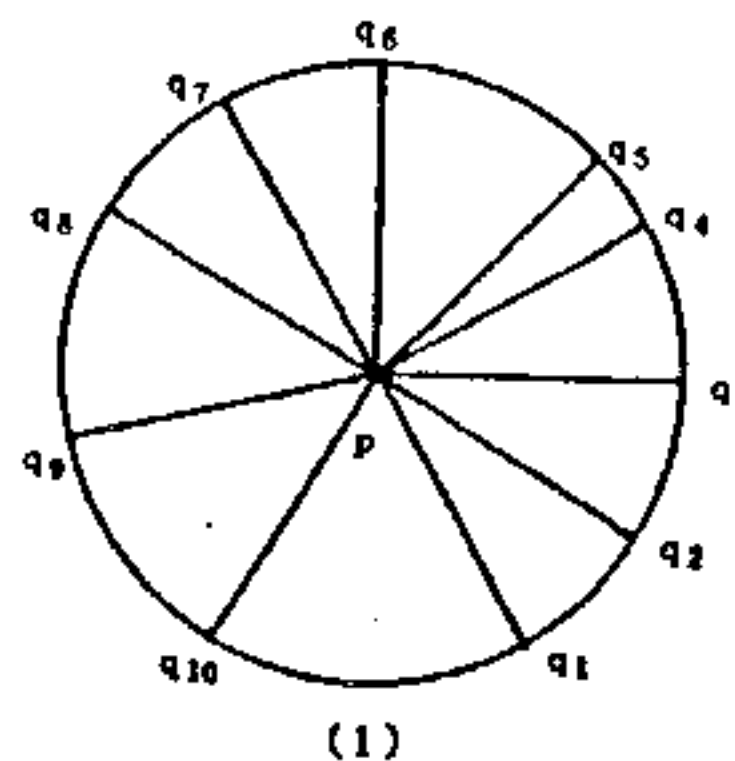
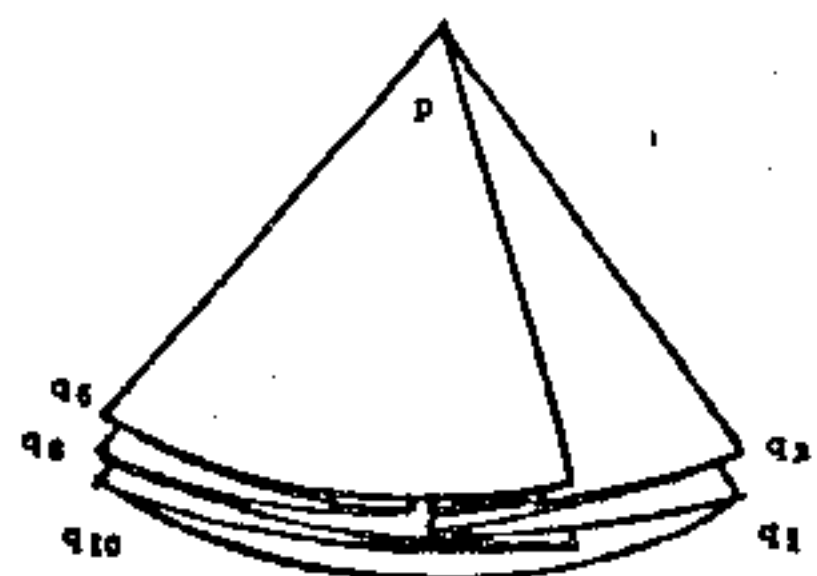


Fig.3.2



(1)

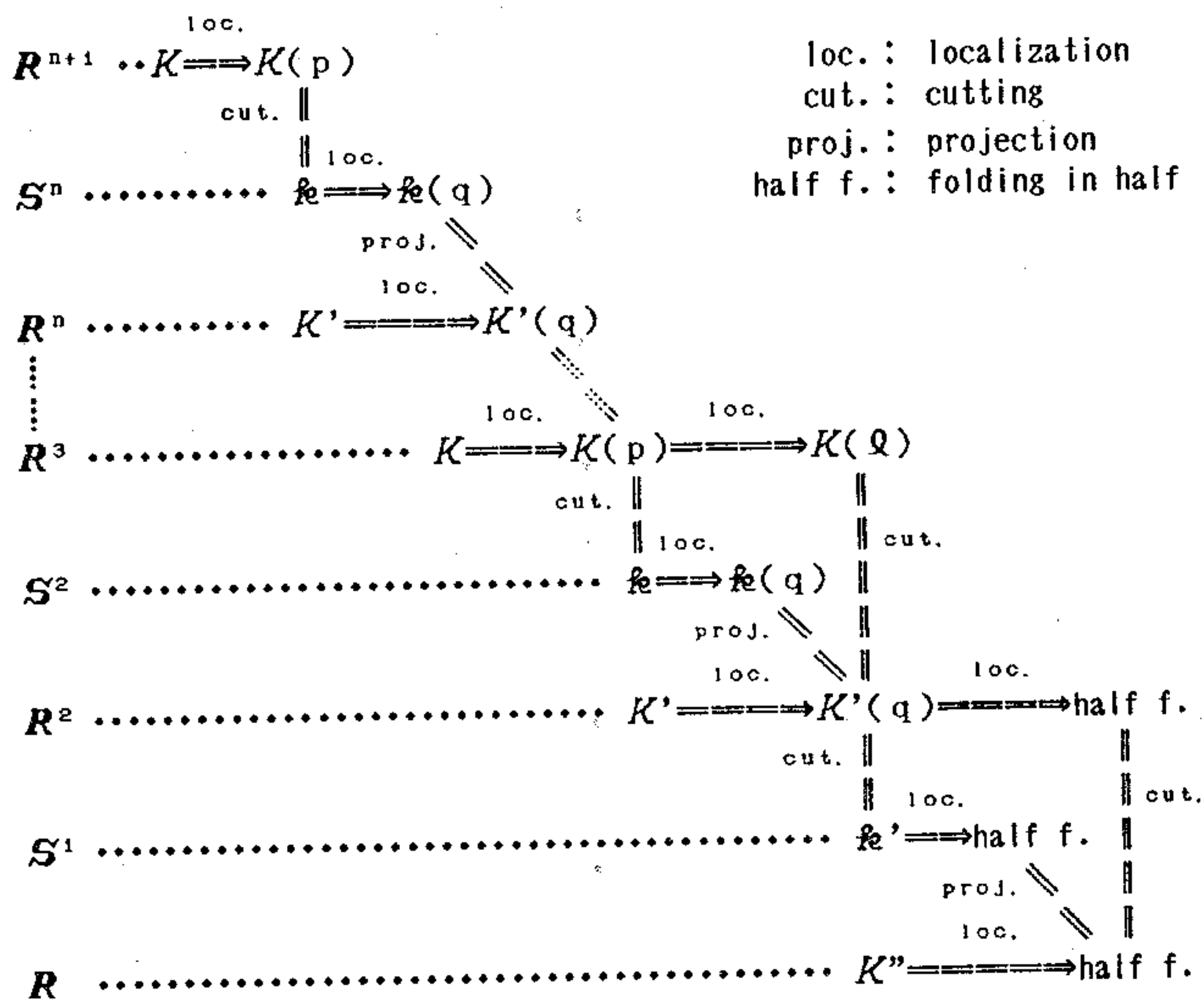


(2)

Fig.3.3 (a flat origami of S^1)

§ 4. Result

We can express the relation between flat origamis of $R, S^1, R^2, S^2, R^3, \dots, R^n, S^n$ and R^{n+1} as folloing:



loc.: localization
 cut.: cutting
 proj.: projection
 half f.: folding in half