

$$R(\gamma) = \mathbf{I}$$

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1. Introduction

Folding a paper is symbolically represented by the relation:

$$R(\gamma) = \mathbf{I}.$$

It is satisfied not only for flat origamis of the plane but also for flat origamis of the 2-sphere, for flat origamis of the high dimensional euclidian spaces and for solid origamis. It is not too much to say that essence of origamis is $R(\gamma) = \mathbf{I}$. In this paper, the author rounds off his study on origami with a view to this relation.

2. Notation

Let $X = \mathbb{R}^2$, $\mathbb{R}^3$ and S^2 be the real euclidian 2-plane, the 3-space and the 2-sphere. We denote their isometric transformation groups by $Isom(X)$. $\mathbf{I}$ is denoted by identical transformation of $Isom(X)$. For a segment (a part of a line) $l \in \mathbb{R}^2$, a face (a part of a plane) $f \in \mathbb{R}^3$ and a great arc (a part of a great circle) $a \in S^2$, we denote by $R(l)$, $R(f)$ and $R(a)$ the reflections respect to l , f and a respectively.

3. The locally flat condition

Let D be a disk in $\mathbb{R}^2$. Let us consider folding it along half lines $l_1, \dots, l_n$ starting from the center P of D (Fig.1). If D is folded in flat then it follows that the alternative sum of the angles between the creases is 180 degrees. We call this condition *the locally flat condition at P*. Since the product of $R(l_i)$ and $R(l_{i+1})$ is a rotation of angle $2\theta_i$, the product $R(l_1)R(l_2)R(l_3)R(l_4) \cdots R(l_{n-1})R(l_n)$ is the rotation of angle $2(\theta_1 + \theta_2 + \cdots + \theta_{n-1}) = 2 \times 180 = 360$ degrees. Thus we have

$$R(l_1) \cdots R(l_n) = \mathbf{I}.$$

4. Flat origami cell decomposition of 2-plane $\mathbb{R}^2$

Definition 4.1 A locally finite cell decomposition (boundary lines are segments, half lines and lines) K of $\mathbb{R}^2$ is called a *flat origami cell decomposition* (development chart) if for an arbitrary closed curve γ in $\mathbb{R}^2$ such that γ does not pass through any vertex (a 0-cell) of K and intersects edges (1-cells) $l_1, \dots, l_n$ of K transversely in this order, $R(\gamma) = R(l_1) \cdots R(l_n) = \mathbf{I}$. We call this relation *the flat condition* (Fig.3).4

For a curve γ which goes around two vertices P_1 and P_2 of a cell decomposition K , let $\gamma = \gamma_1 \gamma_2$ be a decomposition of γ such that γ_i goes around P_i ($i=1, 2$) (Fig.4). Since $R(\gamma) = R(\gamma_1) R(\gamma_2)$, K is a flat origami cell decomposition if and only if the locally flat condition is satisfied for every 0-cell of K .

Definition 4.2 For two 2-cells σ and σ' of a flat origami cell decomposition K , let β be a path starting from σ and ending at σ' such that β does not pass through any 0-cell of K and intersects 1-cells $I_1, \dots, I_n$ of K transversely in this order. We define the transformation $[\sigma, \sigma'] \in \text{Isom}(\mathbb{R}^2)$ as follows:

$$[\sigma, \sigma'] = R(\beta) = R(I_1) \cdots R(I_n).$$

Furthermore the folding map $[\sigma]$ of K toward σ is a piecewise isometric map defined as follows:

$$[\sigma](x) = [\sigma, \sigma'](x) \quad \text{for } x \in \sigma' \in K.$$

There are many paths starting from σ and ending at σ' as in Fig.5. But $[\sigma, \sigma'] = R(\beta)$ does not depend on the path β . Let β and β' be paths starting from σ and ending at σ' . Since combination of β' and inverse β^{-1} of β is a closed curve, $\mathbf{1} = R(\beta' \beta^{-1}) = R(\beta') R(\beta^{-1}) = R(\beta') \{R(\beta)\}^{-1}$. Thus we have $R(\beta) = R(\beta')$. This means that σ' moves on the same position by two actions $R(\beta)$ and $R(\beta')$ when we fold the plane. Therefore the flat condition guarantees that paper is not broken by folding.

5. Flat origami cell decompositions of the 2-sphere S^2

Let us consider folding the 2-sphere S^2 like a balloon. *Flat folding(origami)* of the sphere is folding on itself. If we liken the sphere to the empty earth, it is folding the empty earth along the equator and moving the northern hemisphere on the southern hemisphere (Fig.5,6). Thus we call folding the sphere by *flat folding* if all face, divided by creases (great arcs), of the former sphere move on the sphere after folding.

Another folding of a flat origami (Fig.7) gives a new one (Fig.8). Fig.9 is its development chart. Since the local structure of sphere and of the 2-plane are same, creases satisfy the *locally flat condition*. So we have

Definition 5.1 A locally finite cell decomposition (boundary lines are great arcs) k of S^2 is called a *flat origami cell decomposition* (development chart) if the *locally flat condition* is satisfied for any vertex (0-cell) of k .

The above definition is expressed as follow:

Definition 5.2 A locally finite cell decomposition k of S^2 is called a *flat origami cell decomposition* if for an arbitrary closed curve γ in S^2 such that γ does not pass through any 0-cell (a vertex) of k and intersects 1-cells (great arcs) $a_1, \dots, a_n$ of k transversely in this order,

$$R(\gamma) = R(a_1) \cdots R(a_n) = \mathbf{1} \in \text{Isom}(S^2)$$

(Fig.10).

In the 2-sphere S^2 , flat folding is also represented by the relation: $R(\gamma) = \mathbf{1}$. So we call it the

flat condition in the case of 2-sphere S^2 .

6. Flat origami cell decompositions of 3-space R^3

We consider a reflection $R(f) \in \text{Isom}(R^3)$ with mirror f which the plane including the cone generated by an great arc a of S^2 and its center O . Restriction of $R(f)$ to S^2 is $R(a)$. Therefore the flat condition: $R(\gamma) = R(a_1) \cdots R(a_n) = \mathbf{1} \in \text{Isom}(S^2)$ is restriction of the relation: $R(\gamma) = R(f_1) \cdots R(f_n) = \mathbf{1} \in \text{Isom}(R^3)$ to S^2 , where each f_i is the plane generated by a great arc a_i of S^2 and its center O . Embedding S^2 in R^3 , we find that a orientated closed curve γ in S^2 passes through faces(planes) $f_1, \dots, f_n$ in R^3 . Thus we get

Definition 6.1 A locally finite cell decomposition K of R^3 is called a *flat origami cell decomposition* if for an arbitrary closed curve γ in R^3 such that γ does not pass through any 1-cell (a segment) of K and passes through 2-cells(faces) $f_1, \dots, f_n$ of K transversely in this order,

$$R(\gamma) = R(f_1) \cdots R(f_n) = \mathbf{1} \in \text{Isom}(R^3).$$

Creases(?) of a flat origami of R^3 are not lines but planes(Fig.12). Fig.13 shows 2-cells of a flat origami cell decomposition K of R^3 . It has only one 0-cell and corresponds to the flat origami cell decomposition as in Fig.1.

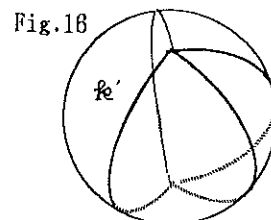
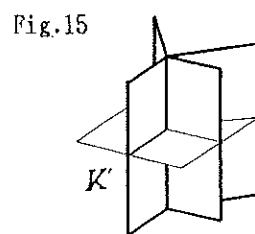
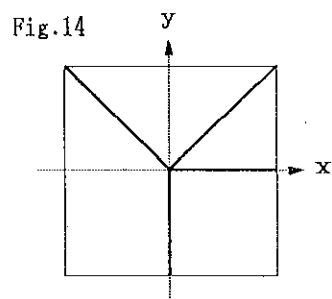
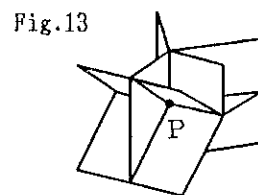
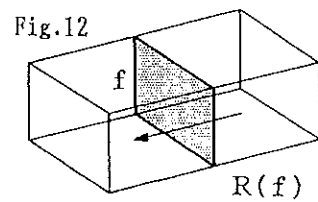
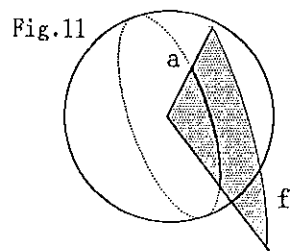
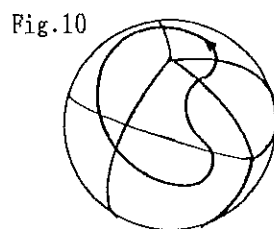
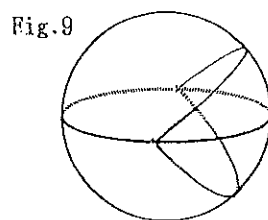
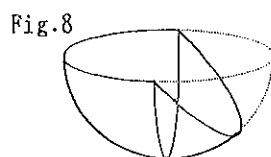
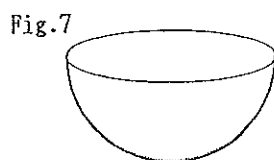
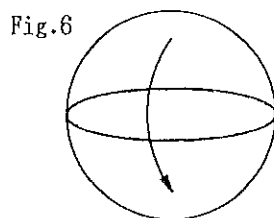
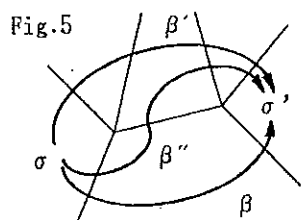
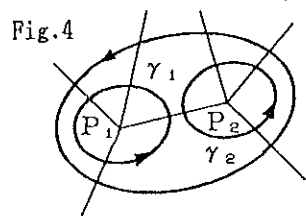
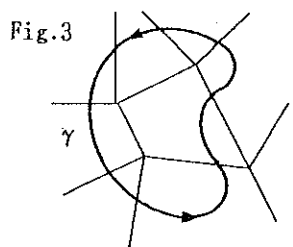
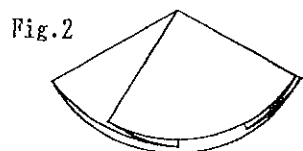
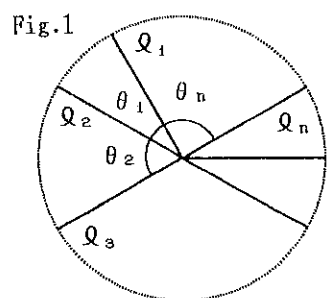
Proposition 6.2 Let K be a cell decomposition of R^3 . We assume that it has only one 0-cell P . Then K is a flat origami cell decomposition of R^3 if and only if for an arbitrary 1-cell m starting at P of K the alternative sum of the angles between 2-cells including m is 180 degrees.

For a cell decomposition K of R^3 such that it has only one 0-cell P and a sphere S with the center P , intersection $k = K \cap S$ is also a cell decomposition of S^2 . By Proposition 6.2, k is a flat origami cell decomposition of S^2 if and only if K is a flat origami cell decomposition of R^3 . This means that the local structure of a flat origami cell decomposition of R^3 is same as the global structure of a flat origami cell decomposition of S^2 . So we have

Proposition 6.3 The local structure of a flat origami cell decomposition of R^3 is same as the global structure of a flat origami cell decomposition of S^2 . Furthermore the local structure of a flat origami cell decomposition of S^2 and of R^2 are the same. And their flat folding are represented by the same relation: $R(\gamma) = \mathbf{1}$.

Example 6.4 A cell decomposition K shown in Fig.14 of the xy -plane is a flat origami cell decomposition of R^2 . If we move K along z -axis then we have a flat origami cell decomposition K' (Fig.15) of R^3 . The relation $R(\gamma) = \mathbf{1}$ for a closed curve γ in xy -plane guarantees that the flat condition is satisfied for K' . Similarly we obtain a flat origami cell decomposition k (Fig.16) of S^2 from K .

Example 6.6 Adding the xy -plane to K' (Ex.6.4), we have a new flat origami cell decomposition K'' (Fig.17) of R^3 . Intersection $k' = K'' \cap S$ is shown in Fig.18. Of course it is a flat origami cell decomposition of S^2 .



Example 6.7 Gluing two flat origami cell decompositions of R^3 is also a flat origami cell decomposition of R^3 (Fig.20,21).

7. Solid origamis

There are two kind of flat foldings along a line. One is mountain folding, the other is valley folding. But we find various solid origamis when we actually fold a paper.

Example 7.1 If we fold a rectangle(Fig.22) along the center line l with rotation of angel $\pi/4$ radian, then we have a solid origami(Fig.23). It is represented by the right-handed screw rotation of angle $\pi/4$ with axis l . We denote it by $R(l, \pi/4)$.

Example 7.2 Two flat folding(Fig.24,25) are represented by $R(l, \pi)$ and $R(l, -\pi)$.

Let us consider the folding map $[\sigma]$ of solid origamis. First we fold the band(Fig.26) along a line l_1 with rotation of angle $\pi/4$ so that the face σ does not move, and then we fold it along a line l_2 with rotation of angle $-\pi/4$. Fig.27 is the finished work. By folding, the faces τ and τ' move on $R(l_1, \pi/4)\tau$ and on $R(l_1, \pi/4)R(l_2, -\pi/4)\tau'$ respectively (Fig.28). Thus we give

Definition 7.3 For two 2-cells σ and σ' of a cell decomposition K of R^2 such that its 1-cells are orientated segments with folding rotation angle θ , let β be a path starting from σ and ending at σ' such that β does not pass through any 0-cell of K and intersects 1-cells $l_1, \dots, l_n$ of K transversely in this order. We define the transformation $[\sigma, \sigma'] \in \text{Isom}(R^2)$ as follows:

$$[\sigma, \sigma'] = R(\beta) = R(l_1, \theta_1) \cdots R(l_n, \theta_n).$$

Furthermore the folding map $[\sigma]$ of K toward σ is a piecewise isometric map defined as follows:

$$[\sigma](x) = [\sigma, \sigma'](x) \quad \text{for } x \in \sigma' \in K.$$

We consider solid folding of the disk D with the center O in the xy -plane such that its creases are segment starting at O . What happen when we fold D along four creases with folding rotation angle $\pi/4$? Fig.31~33 show folding process along the creases such that the path β starting from σ and ending at σ' crosses them(Fig.29). Comparing Fig.30 and Fig.33, we find that $R(\beta)$ is the rotation $R(z\text{-axis}, \pi/2)$. Similarly we find that $R(\beta')$ is the rotation $R(x\text{-axis}, -\pi/2)$. Since the action $R(z\text{-axis}, \pi/2)$ is different from the action $R(z\text{-axis}, \pi/2)$, the face σ' moves to the other places by the actions $R(\beta)$ and $R(\beta')$. This means that the above folding breaks the disk D . Thus we have no-cutting(not-broken) condition:

$$R(\gamma) = R(l_1, \theta_1) \cdots R(l_n, \theta_n) = 1,$$

where γ is a closed curve which goes around one vertex(a 0-cell) of K .

Definition 7.4 A locally finite cell decomposition K of R^2 such that its 1-cells are segments with folding rotation angles is called a solid origami cell decomposition if for an arbitrary closed curve γ in R^2 such that γ goes around one vertex(a 0-cell) of K and intersects edges(1-cells) $l_1, \dots, l_n$ of K transversely in this order, no-cutting condition:

$$R(\gamma) = R(l_1, \theta_1) \cdots R(l_n, \theta_n) = 1$$

Fig.17

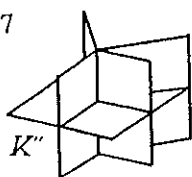


Fig.18

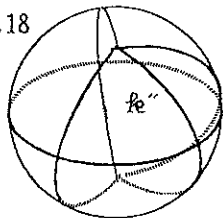


Fig.19

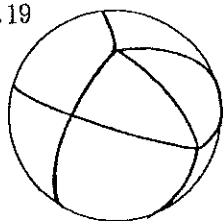


Fig.20

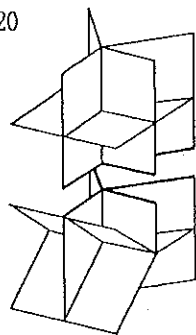


Fig.21

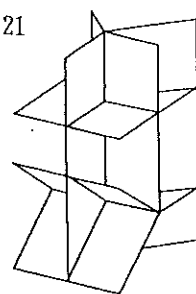


Fig.22

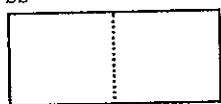


Fig.23

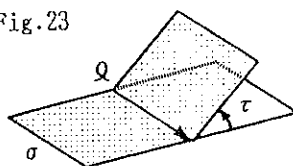


Fig.24

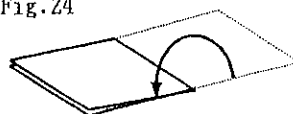


Fig.25

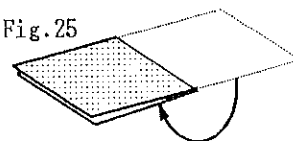


Fig.26

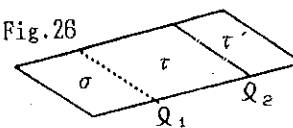


Fig.27

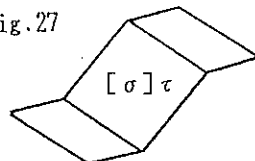


Fig.28

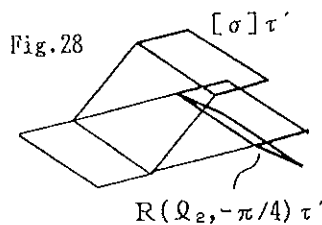


Fig.29

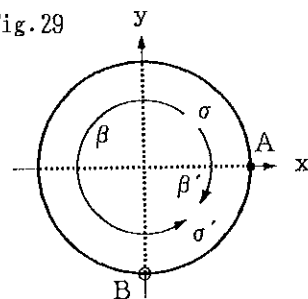


Fig.30

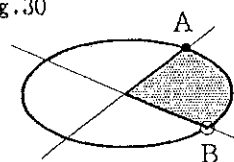


Fig.31

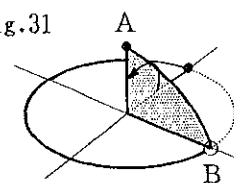


Fig.32

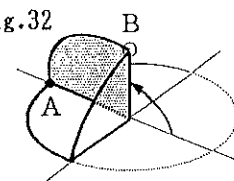


Fig.33

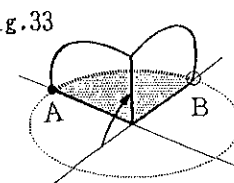
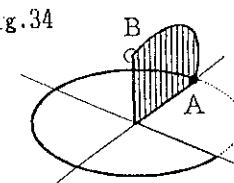


Fig.34



is satisfied, where each 1-cell(segment) l consists of two arrow(orientated segments) and they meet the center of the segment l and γ crosses arrows whose orientations are from left to right.

Complicatedness of the above definition is due to explicit definition of rotations in the 3-space R^3 . Fig.36 shows realization of a *solid origami cell decomposition* K (Fig.35). As above mention, flat folding $R(l)$ is a special case of *solid folding* $R(l, \theta)$ i.e. $R(l) = R(l, \pm \pi)$. So the *flat condition*: $R(\gamma) = R(l_1) \cdots R(l_n) = \mathbf{1} \in \text{Isom}(R^2)$ is restriction of the *no-cutting condition*: $R(\gamma) = R(l_1, \pm \pi) \cdots R(l_n, \pm \pi) = \mathbf{1} \in \text{Isom}(R^3)$ to R^2 .

8. The no-cutting condition and non-penetrability of a paper

Solid origamis and flat origamis are symbolically represented by the *no-cutting condition* $R(\gamma) = R(l_1, \theta_1) \cdots R(l_n, \theta_n) = \mathbf{1}$. It seems that *mountain folding* and *valley folding* are also held by the *no-cutting condition*.

We compare two *solid origami cell decompositions* K_1 and K_2 of the disk D such that those 1-cells are x-axis and y-axis. The 1-cells l_1, l_2 and l_3 of K_1 are valley creases. The 1-cell l_4 of K_1 are mountain creases(Fig.37). All 1-cells of K_2 are valley creases(Fig.38). Though they satisfy the *no-cutting condition* $R(\gamma) = R(l_1, \pi)R(l_2, \pi)R(l_3, \pi)R(l_4, -\pi) = \mathbf{1}$ and $R(\gamma') = R(l_1, \pi)R(l_2, \pi)R(l_3, \pi)R(l_4, \pi) = \mathbf{1}$ respectively, we can actually fold K_1 only. Therefore the *no-cutting condition* does not guarantee existence of realization of *solid origami cell decomposition*. Indeed, $R(l_4, -\pi) = R(l_4, \pi)$ means that *mountain folding* and *valley folding* are not distinguished by use of the rotation $R(l, \theta)$. The *no-cutting condition* is only necessary condition of real folding. In order to express folding process, we need a family of *solid origami cell decompositions*.

Definition 8.1 For a *solid origami cell decomposition* K_1 of R^2 , a family of *solid origami cell decompositions* of $R^2: \{K_t; 0 \leq t \leq 1\}$ is called a *mathematical folding process* of K_1 and θ is called a *folding process function*, if the following conditions are satisfied,

- (1) As cell decompositions of R^2 , $K_t = K_1$ for $0 \leq t \leq 1$,
- (2) For each 1-cell, its rotation angle θ is a continuous function: $[0, 1] \rightarrow [-\pi, \pi]$,
- (3) $\theta(0) = 0$.

Example 8.2 Fig.41~43 show a folding process of the *solid origami cell decomposition* as in Fig.40. *Folding process functions* θ_i are

$$\begin{aligned} \theta_1(t) = \theta_4(t) &= \begin{cases} 3t\pi & (0 \leq t \leq \frac{1}{3}) \\ \pi & (\frac{1}{3} \leq t \leq 1) \end{cases} \\ \theta_2(t) = -\theta_5(t) &= \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ -3(t - \frac{1}{3})\pi & (\frac{1}{3} \leq t \leq \frac{2}{3}) \\ -\pi & (\frac{2}{3} \leq t \leq 1) \end{cases} \\ \theta_3(t) = -\theta_6(t) &= \begin{cases} 0 & (0 \leq t \leq \frac{2}{3}) \\ 3(t - \frac{2}{3})\pi & (\frac{2}{3} \leq t \leq 1) \end{cases} \end{aligned}$$

Fig.35

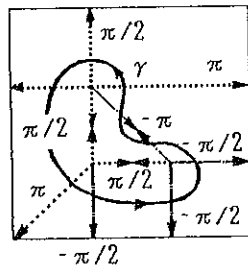


Fig.36

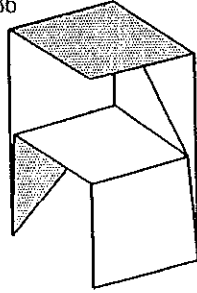


Fig.37

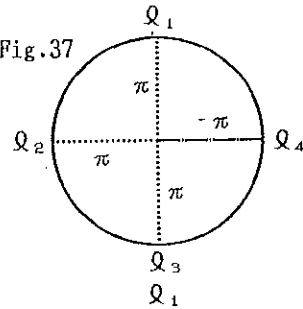


Fig.38

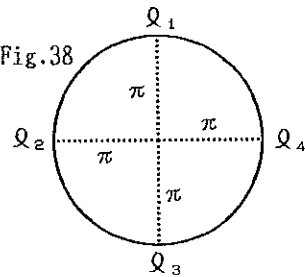


Fig.39



Fig.40

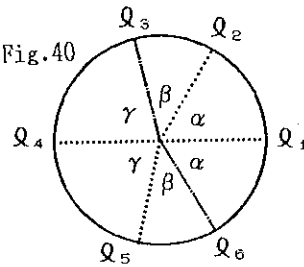


Fig.41

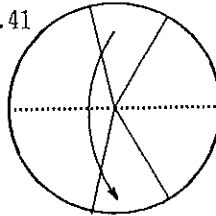


Fig.42

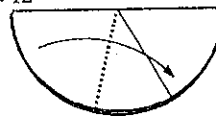


Fig.43

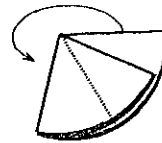


Fig.44

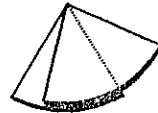


Fig.45

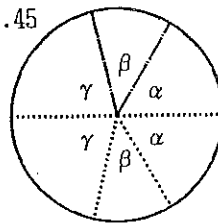


Fig.46

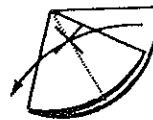


Fig.47

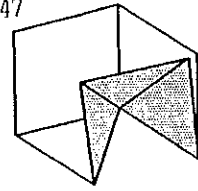
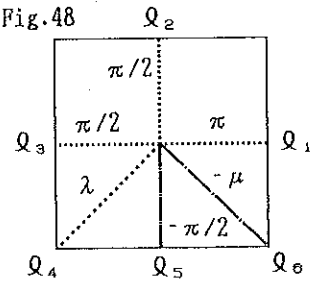


Fig.48



The pair of a solid origami cell decomposition K and folding process functions θ_i i.e. a mathematical folding process expresses well actual folding process. But it is not perfect.

Example 8.3 The cell decomposition K' (Fig.44) is a solid(flat) origami cell decomposition, where the dotted lines are valley creases, the chained lines are mountain creases and $\beta < \alpha \leq \gamma$. If we replace the folding process functions θ_3 and θ_6 of Ex.8.2 to the following:

$$\theta_3(t) = -\theta_6(t) = \begin{cases} 0 & (0 \leq t \leq \frac{2}{3}) \\ -3(t - \frac{2}{3})\pi & (\frac{2}{3} \leq t \leq 1), \end{cases}$$

$\{K'; 0 \leq t \leq 1\}$ is a mathematical folding process of $K'=K'$. But we can not fold it actually according to this mathematical folding process. Because the paper disturbs folding by itself for $t \geq \frac{2}{3}$, as in Fig.45.

Therefore we find again that existence of a mathematical folding process does not guarantee real folding. And non-penetrability of a paper is the reason why existence of a mathematical folding process is only a necessary condition.

9. Appendix

K.Kasahara[2] calls solid origamis by "hankai-ori". We show that a Kasahara's hankai-ori named by "A dislocation of cube" (Fig.46) satisfies the no-cutting condition. Put $R(x\text{-axis}, \theta) = x(\theta)$ and $R(z\text{-axis}, \theta) = z(\theta)$. For the closed curve γ shown in Fig.47, $R(\gamma) = R(I_1, \pi)R(I_2, \pi/2)R(I_3, \pi/2)R(I_4, \alpha)R(I_5, -\pi/2)R(I_6, -\beta) = \mathbf{1}$. Since $R(I_1, \pi) = x(\pi)$, $R(I_2, \pi/2) = z(\pi/2)x(\pi/2)z(-\pi/2)$, $R(I_3, \pi/2) = x(-\pi/2)$, $R(I_4, \alpha) = z(-3\pi/4)x(\alpha)z(3\pi/4)$, $R(I_5, -\pi/2) = z(-\pi/2)x(-\pi/2)z(\pi/2)$, $R(I_6, -\beta) = z(-\pi/4)z(-\beta)z(\pi/4)$, $\cos \alpha = -1/3$, $\sin \alpha = 2\sqrt{2}/3$, $\cos \beta = 1/3$ and $\sin \beta = 2\sqrt{2}/3$. These rotations are represented by the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}.$$

If we calculate product of these matrices in this order, we find that it is the identical matrix. What does Kasahara think that "A dislocation of cube" is represented by the following relation?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ 2/3 & -2/3 & -1/3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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