Solving Cubics With Creases: The Work of Beloch and Lill

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Abstract. Margharita P. Beloch was the first person, in 1936, to realize that origami (paper-folding) constructions can solve general cubic equations and thus are more powerful than straightedge and compass constructions. We present her proof. In doing this we use a delightful (and mostly forgotten?) geometric method due to Eduard Lill for finding the real roots of polynomial equations.

1. INTRODUCTION. There are many aspects to the mathematics of origami, or paper folding. One may study combinatorial properties that emerge from folded paper. One can study origami as mappings from the Euclidean plane into three-dimensional space that have certain properties. But the oldest way to study origami mathematically is as a method for geometric constructions. The idea is to take a piece of paper and fold it, making a straight crease line. Then we unfold the paper and make another crease line. In doing this we start locating points of intersection of our crease lines and thus can try to construct geometric figures, like an equilateral triangle or the angle bisector between two lines.

This is similar, of course, to straightedge and compass constructions, except it is not immediately clear that the circle-making power of a compass could be duplicated by origami since we can only make straight crease lines. A paper-folding skeptic might thus be surprised to learn that origami constructions are actually more powerful than those made by straightedge and compass. Origami can trisect angles (see [**10, 12, 14, 22**]) and double cubes (see [**22, 23**]), as well as solve general cubic equations (see [**1, 6, 9**]).

But who was the first person to discover the full power of paper folding as a geometric construction tool? The credit goes to an Italian mathematician named Margharita Piazolla Beloch in the 1930s [**4**]. Given this, it is perhaps more than a little embarassing that numerous researchers since [**1, 3, 6, 9, 22**], including the author [**10**], have failed to cite Beloch's ground-breaking work. (Huzita, Scimemi [**13**], and Justin [**14**] are notable exceptions.)

In this paper, we present Beloch's proof that paper folding can solve arbitrary cubic equations and thus solve the classic problems of angle trisection and doubling the cube. At the same time, we will encounter a marvelous geometric method for finding real roots of arbitrary polynomials due to Eduard Lill [**19**]. We finish with a more extensive accounting of the history of origami geometric constructions, arguing that Beloch was, indeed, the first person to discover the full power of normal paper folding.

2. BELOCH'S SQUARE. Like straightedge and compass constructions, any paper-folding construction can be described as a sequence of elementary folding moves, or axioms, as some call them. These basic moves can be classified by enumerating all of the possible ways a single, straight crease line can be made by aligning given points and lines to other points and lines already made on your paper [**17**]. Some examples of these basic folding moves are:

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O1: Given two points P_1 and P_2, we can make a crease line that places P_1 onto P_2 when folded.

O2: Given a line l and a point P not on l, we can make a crease line that passes through P and is perpendicular to l.

For more information on these basic moves see [**13**] and [**17**]. Note, however, that the above two basic moves can also be done by a straightedge and compass. The one basic folding move which sets origami apart from straightedge and compass constructions is the following:

The Beloch Fold. Given two points P_1 and P_1 and two lines l_1 and l_2 we can, whenever possible, make a single fold that places P_1 onto l_1 and P_2 onto l_2 simultaneously. (See Figure 1.)

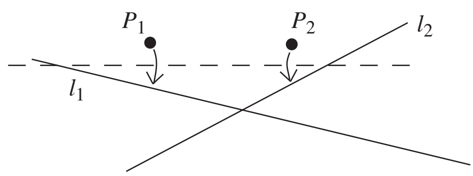

Figure 1. The Beloch origami fold.

One way to see what this fold is doing is to consider one of the point-line pairs. If we fold a point P to a line l, the resulting crease line will be tangent to the parabola with focus P and directrix l (the equidistant set from P and l). This can be demonstrated by the following activity: Take a piece of paper, draw a point P on it, and let the bottom edge of the paper be the line l. Then fold P to l over and over again. An easy way to do this is to pick a point on l and fold it up to P, unfold, then pick a new point on l and fold it to P, and repeat. After a diverse sampling of creases are made, the outline of a parabola seems to emerge. Or, more precisely, the envelope of the crease lines seems to be a parabola. (See Figure 2(a).) A proof of this can be established as follows: After folding a point P' on l to P, draw a line perpendicular to the folded image of l, on the folded flap of paper from P to the crease line, as in Figure 2(b). If X is the point where this drawn line intersects the crease line, then we see when unfolding the paper that the point X is equidistant from the point P and the line l. (See Figure 2(c).) Any other point on the crease line will be equidistant from P and P' and thus will not have the same distance to the line l. Therefore the crease line is tangent to the parabola with focus P and directrix l.

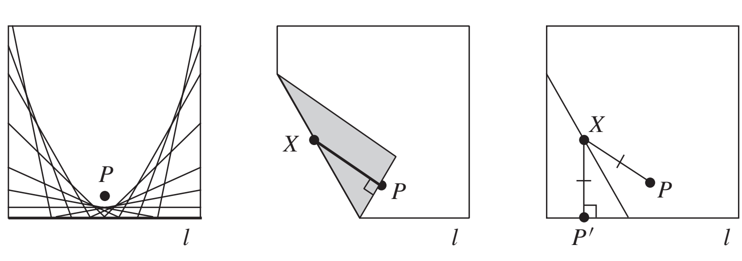

Figure 2. Folding a point to a line creates tangents to a parabola.

In other words, folding a point to a line can be thought of as locating a point on a certain parabola, which means that this is equivalent to solving a quadratic equation.

The Beloch fold can then be interpreted thusly: Folding P_1 to l_1 will make the crease be tangent to the parabola with focus P_1 and directrix l_1, and folding P_2 to l_2 will make the crease be tangent to the P_2-focused and l_2-directrixed parabola. In other words, this origami fold finds a common tangent to two parabolas.

Now, two parabolas drawn in the plane can have at most three different common tangents (for example, see Figure 3), suggesting that this origami fold is equivalent to solving a cubic equation. Straightedge and compass constructions, on the other hand, can only solve general quadratic equations.

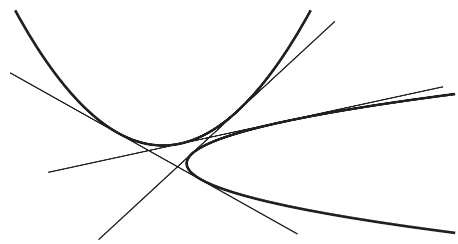

Figure 3. Two parabolas drawn in the plane can have at most three common tangents.

Theoretically, we could end this paper right here, satisfied in the knowledge that origami can solve cubic equations. (Specifically, on the projective plane, finding common tangents to two parabolas is the dual problem to finding intersections of conics, which allows general cubic solutions to be constructed. See [**1**] and [**28**].) But Beloch provides a constructive proof. She considers the following problem:

The Beloch Square. Given two points A and B and two lines r and s in the plane, construct a square $WXYZ$ with two adjacent corners X and Y lying on r and s, respectively, and the sides WX and YZ, or their extensions, passing through A and B, respectively. (See Figure 4.)

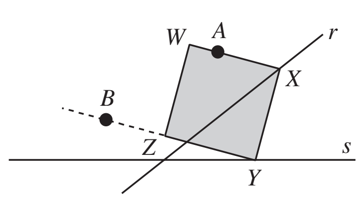

Figure 4. The Beloch square.

Amazingly, this one construction problem captures everything we need not only to construct $\sqrt[3]{2}$ (thus solving one of the classic Greek construction problems, that of doubling the volume of a cube) but also to solve arbitrary cubic equations. Furthermore, this problem is readily solved via origami!

Here is how: We are given points A and B and lines r and s. Compute the perpendicular distance from A to r and create a new line r' which is this same distance from and parallel to r, so that r lies between A and r'. Do the same with B and s to construct a line s'. (See Figure 5, left.) Note that these lines r' and s' can be constructed easily via paper folding by, say, folding along r, marking where A lands under this fold, and then making a sequence of perpendicular folds O2 described above. (The details of this are left as an exercise.)

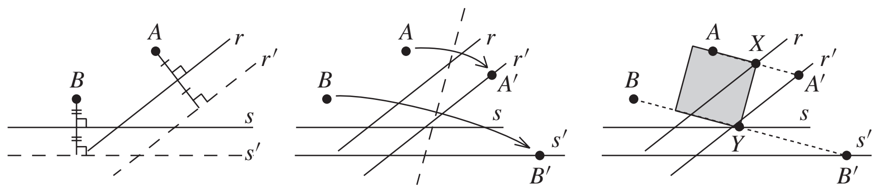

Figure 5. Constructing the Beloch Square using origami.

We then perform the Beloch fold, folding A onto r' and B onto s' simultaneously. (See Figure 5, center.) This will fold A to a point A' on r' and B onto a point B' on s'. The crease made from this fold will be the perpendicular bisector of the segments AA' and BB'. Therefore, if we let X and Y be the midpoints of AA' and BB', respectively, we have that X lies on r and Y lies on s because of the way in which r' and s' were constructed. The segment XY can then be one side of our Beloch square, and since AX and BY are perpendicular to XY, we have that A and B are on opposite sides, or extensions of sides, of this square.

3. CONSTRUCTING $\sqrt[3]{2}$. Next we will see how Beloch's square allowed her to construct the cube root of two. (Actually, what follows is her construction set on coordinate axes.) Let us take r to be the y-axis and s to be the x-axis of the plane. Let $A = (-1, 0)$ and $B = (0, -2)$. Then we construct the lines r' to be $x = 1$ and s' to be $y = 2$. Folding A onto r' and B onto s' using the Beloch fold will make a crease which crosses r at a point X and s at a point Y. Consulting Figure 6, if we let O be the origin, then notice that OAX, OXY, and OBY are all similar right triangles. This follows from the fact that XY is perpendicular to AA' and BB'.

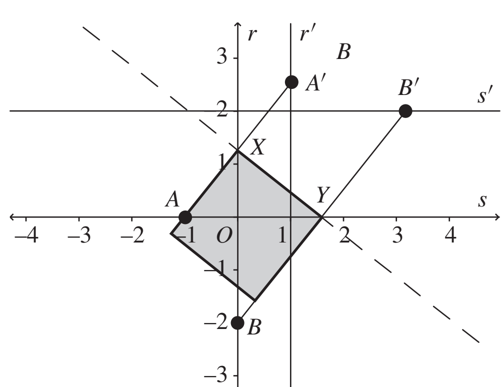

Figure 6. Beloch's origami construction of the cube root of two.

Therefore, we have $|OX|/|OA| = |OY|/|OX| = |OB|/|OY|$, where $|\cdot|$ denotes the length of the segment. Filling in $|OA| = 1$ and $|OB| = 2$ gives us $|OX| = |OY|/|OX| =$

2/|OY|. Finally, compute

$$|OX|^3 = |OX| \cdot \frac{|OY|}{|OX|} \cdot \frac{2}{|OY|} = 2,$$

and so $X = (0, \sqrt[3]{2})$.

This construction is essentially the same as the one independently discovered by Martin [**22**] fifty years later, although Martin takes $B = (0, -k)$ so as to construct $X = (0, \sqrt[3]{k})$.

4. SOLVING CUBIC EQUATIONS. Beloch goes on to describe how her square construction leads to a paper-folding method for finding real roots of arbitrary cubic equations. For this she refers to "the famous procedure of Lill for the graphical resolution of equations of third degree" [**4**]. This "Lill's method" does not seem to be as famous now as it was in the 1930s. She is referring to an 1867 paper [**19**] by an Austrian engineer named Eduard Lill. Felix Klein describes the cubic case of Lill's method in his 1926 book [**16**, p. 267], and he refers to it as well known as well. The general method was described in a paper by Riaz in this MONTHLY in 1962 [**24**], but since more recent citations of Lill's method are rare, we will reproduce this elegant method here.

Suppose we are given a polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$ with real coefficients and we would like to locate a real root of $f(x)$, if one exists. Lill suggests doing this geometrically by creating a path in the plane based on the coefficients of $f(x)$.

Imagine a turtle is sitting at the origin O and facing in the direction of the positive x-axis. (Note that Lill did not use a turtle in his original exposition, but the analogy to modern turtle graphics makes the metaphor especially apt.) The turtle will walk along the positive x-axis a distance equal to the coefficient a_n. Then the turtle will turn 90° counterclockwise and walk a distance equal to the next coefficient a_{n-1}. The turtle will then turn again and repeat this process until ending at a point T after traveling a distance a_0 in some direction. If any of the coefficients are negative then the turtle will walk backwards and that side of the turtle path will be considered to have negative length. (E.g., in Figure 7(b) the sides marked a_3, a_2, and a_0 all have negative length.) If any of the coefficients are zero then the turtle will still turn but walk a distance of zero.

After the turtle has made this path we will position ourselves at O and then attempt to "shoot" the turtle at T in the following way: We imagine that we are living in a

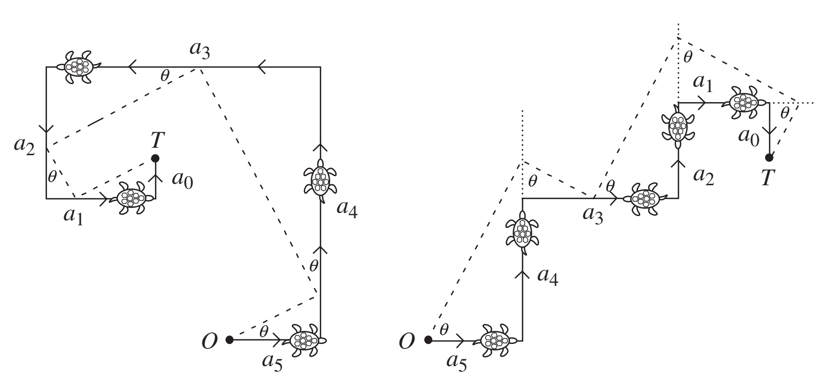

Figure 7. Lill's method turtle and bullet paths (a) for a quintic with all coefficients positive and (b) a quintic with $a_3, a_2, a_0 < 0$ and $a_5, a_4, a_1 > 0$.

universe where bullets bounce off walls at 90° angles. We fire a bullet from O at the line containing the turtle path segment of length a_{n-1}. This bullet will ricochet off this line at a right angle, then ricochet off the line containing the side of length a_{n-2}, and so on. Note that the act of 90° ricocheting is ambiguous, since sometimes we want it to bounce off the line on the same side as the bullet's approach, while other times we want it to bounce through the line (but still at a right angle). In all cases, we make sure to choose the option that will allow the bullet to actually hit the next side of the turtle path. See Figure 7. If we are able to "hit" the turtle in this way, then the bullet path will have n sides and our turtle path $n + 1$ sides.

Let θ be the angle that the first part of our bullet path makes with the x-axis (which contains the a_n side of the turtle path), assuming that we are actually able to hit the turtle.

Claim. $x = -\tan\theta$ is a root of $f(x)$.

Our proof of this will assume that all our coefficients are positive, and so our turtle and bullet paths will be as in Figure 7(a). The cases with coefficients negative or zero are left as an exercise.

Notice that the sides of the bullet path are the hypotenuses of a sequence of similar right triangles whose legs lie along the turtle path. Let y_k be the length of the side opposite the angle θ in the triangle whose side adjacent to θ is part of the segment of length a_k. Then we get

$$\begin{aligned}
y_n &= (\tan\theta)a_n = -xa_n \\
y_{n-1} &= (\tan\theta)(a_{n-1} - (-xa_n)) = -x(a_{n-1} + xa_n) \\
y_{n-2} &= (\tan\theta)(a_{n-2} - (-x(a_{n-1} + xa_n))) = -x(a_{n-2} + x(a_{n-1} + xa_n)) \\
&\vdots \\
y_1 &= -x(a_1 + x(a_2 + \cdots + x(a_{n-2} + x(a_{n-1} + xa_n))\cdots)).
\end{aligned}$$

But $y_1 = a_0$. Equating these two values for y_1 and simplifying gives us $f(x) = 0$. If no value of θ will allow us to hit the turtle, then $f(x)$ must have no real roots.

Lill's method is nothing short of amazing, and it gets better. The bullet path turns out to be similar (in the geometric sense) to the turtle path one would obtain from the polynomial $f(x)$ with $(x + \tan\theta)$ factored out. For example, Riaz [**24**] demonstrates this with the polynomial $x^3 - 7x - 6$. This has three real roots, and each one corresponds to a different angle θ to shoot the turtle. If we pick one, say the one which gives the root $x = 3$, then the bullet path will be a rotated dilation of the turtle path for the polynomial $x^2 + 3x + 2$. (If the reader has dynamic geometry software available, it can be used to demonstrate Lill's method very convincingly, and this is highly recommended.)

Beloch's stroke of brilliance in paper-folding constructions was in seeing that Lill's method in the cubic case is just an application of her square construction. Indeed, in the cubic case our turtle path for $a_3x^3 + a_2x^2 + a_1x + a_0$ will have four sides, so our bullet path will have three sides. If we think of O as the point A, and T as the point B, and we think of the lines containing the a_2-side and the a_1-side as the lines r and s, respectively, then a Beloch square with adjacent corners on r and s and opposite sides passing through O and T will give us a bullet path to shoot this turtle. (See Figure 8.) Therefore paper folding can be used to perform Lill's method in the cubic case and thus solve general polynomials of degree three.

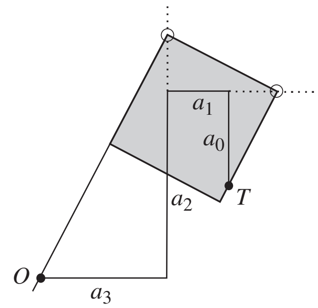

Figure 8. Lill's method, cubic case, is really constructing a Beloch square.

5. ORIGAMI GEOMETRY HISTORY AND FUTURE.

The idea of using paper folding as a geometric construction tool has unknown origins. Ancient Japanese *sangaku* (mathematical problems painted on wooden tablets and hung in shrines, circa 1600–1890) have been found that depict paper-folding geometry problems, indicating that the Japanese have mathematical as well as religious and artistic traditions in origami [**8**]. But the first known treatise on paper-folding constructions is T. Sundara Row's book *Geometric Exercises in Paper Folding* [**25**], first published in 1893. This book was mentioned by Felix Klein in one of his popular math books of the time [**15**], and this seems to have helped popularize paper-folding geometry. Beloch herself states that Klein was the first to attract students to Row's book with his "autorevole giudizo" (authoritative judgement) [**4**]. However, Row does not try to classify the basic origami moves (axioms). He defines paper folding very broadly, employing folding moves as needed that place points and lines onto previously constructed points and lines, and he does not mention or make use of anything like Beloch's fold. In fact, Row mistakenly claims that it is impossible to construct the cube root of two exactly with paper folding [**25**, Section 112].

Row and Klein seem to have sparked a general interest in the geometry of paper folding in the early 1900s. A number of papers appeared around that time focusing on solving quadratic equations elegantly via origami, all of which cite Row as a primary influence. For examples, see Lotka's 1907 *School Science and Mathematics* paper [**21**] and Rupp's 1924 paper in this MONTHLY [**26**].

In 1930 Giovanni Vacca wrote an article [**27**] in the Italian journal *Periodico di Mathematiche* whose title translates as "On the folding of paper applied to geometry." In it Vacca briefly describes the history of paper folding, tracing it back to Chinese and Japanese origins (although some of these are clearly speculative), and then proceeds to describe everything that is known about the connections between origami and geometry at the time, including references to Row, the influential educator Friedrich Froebel, and others. He summarizes how origami can solve quadratic equations, but makes no mention about whether or not origami could solve cubics. Vacca does not, however, repeat Row's claim that origami cannot construct cube roots.

This set the stage for Margherita Piazzolla Beloch, an algebraic geometer at the University of Ferrara, Italy. She was born in 1879 and was the daughter of the University of Rome's renowned historian Karl Julius Beloch. She received her doctorate in mathematics in 1908 at the University of Rome under Guido Castelnuovo. She held positions at the University of Pavia and Palermo, working with Michele de Franchis. In 1927 she was made Chair of Geometry at the University of Ferrara, where she remained until her retirement in 1955. While her primary research was in algebraic geometry, many of her papers were on the application of geometry to *photogram-*

metry, the study of computing three-dimensional image data from photographs, such as x-ray or aerial images. Beloch passed away while living in Rome in 1976 [**20**].

In 1936 Beloch published "Sul metodo del ripiegamento della carta per la risoluzione dei problemi geometrici" (On the method of paper folding for the resolution of geometric problems), in the journal *Periodico di Mathematiche* [**4**]. She describes this paper as an extract from a mathematics course she taught at Ferrara during the academic year 1933–34. This, together with Vacca's 1930 paper, seems to be strong evidence that Beloch was the first to discover that origami can find common tangents to two parabolas by folding two points to two lines simultaneously and thus solve general cubic equations. Beloch was also quick to note that since solving quartics can be reduced to solving cubics and quadratics, we know that origami can find real roots of quartic equations.

Since then it has been demonstrated that the Beloch fold is the most complicated paper-folding move possible [**17**]. By this we mean that if one tries to write a list of all possible origami moves (like O1 and O2 described earlier) that only produce a single, straight crease line, then no other such origami move will give us more algebraic power than the Beloch fold. In other words, Beloch's work does, in fact, determine the constructible limit of normal paper folding.

To clarify, however, this work concerns only straight-crease, one-fold-at-a-time origami. Other directions in origami constructions can and have been explored. Folding curved creases (not straight lines) is possible, although difficult [**7**], and spoils the construction game completely by allowing transcendentals like π to be constructed [**11**]. Also, Robert Lang has demonstrated that if we allow ourselves to make *simultaneous creases*, i.e., an origami move that produces more than one crease line, made in unison (such origami moves are called *multifolds*), then arbitrary angle quintisections can be performed [**18**]. In fact, Alperin and Lang have recently shown that if three simultaneous creases are allowed, then arbitrary quintics can be solved [**2**]. They use the quintic case of Lill's method to demonstrate that this can be done in theory, although actually performing such a complex fold seems physically impossible to do in general. Furthermore, Chow and Fan argue [**5**] that roots of polynomials of arbitrary degree can be found if any number of simultaneous folds are allowed. Nonetheless, Beloch deserves the credit for first discovering the geometric limits of origami that mere mortals are able to perform.

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