Counting and 3-Edge-Coloring Spherical Buckyballs

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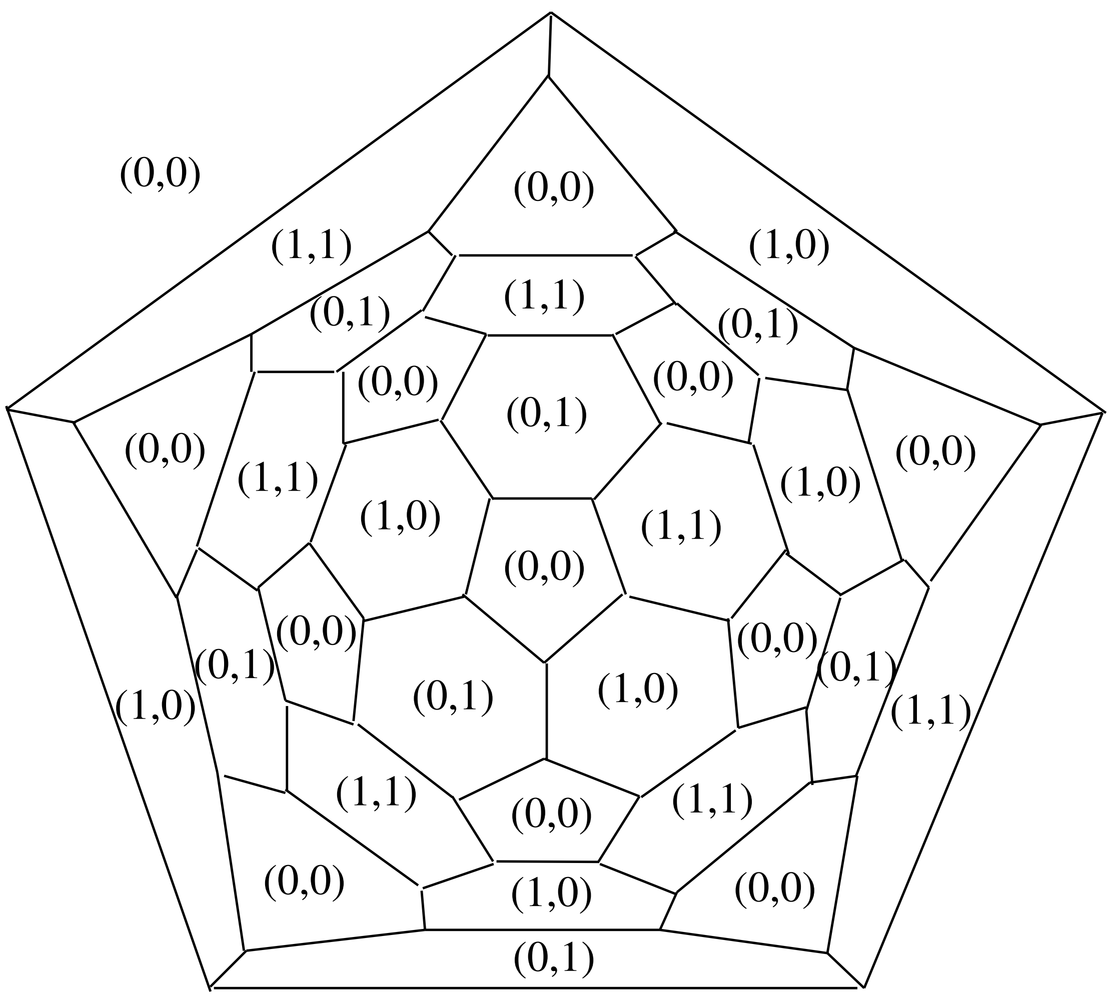
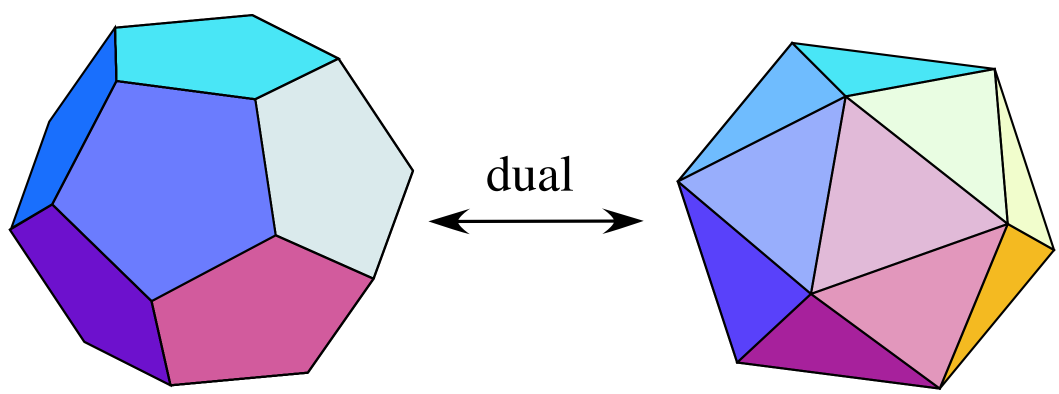


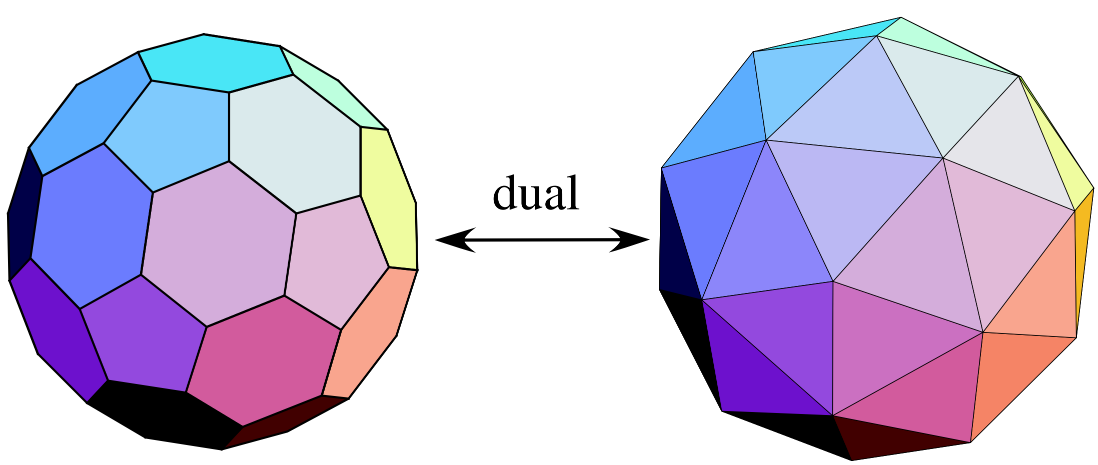
What Buckyballs are made of

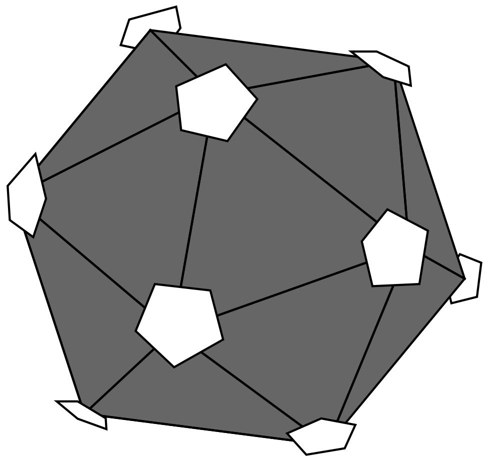
Definition: A **Buckyball** is a polyhedron with only pentagon and hexagon faces and all vertices of degree 3.

Fact: All Buckyballs have exactly 12 pentagons.

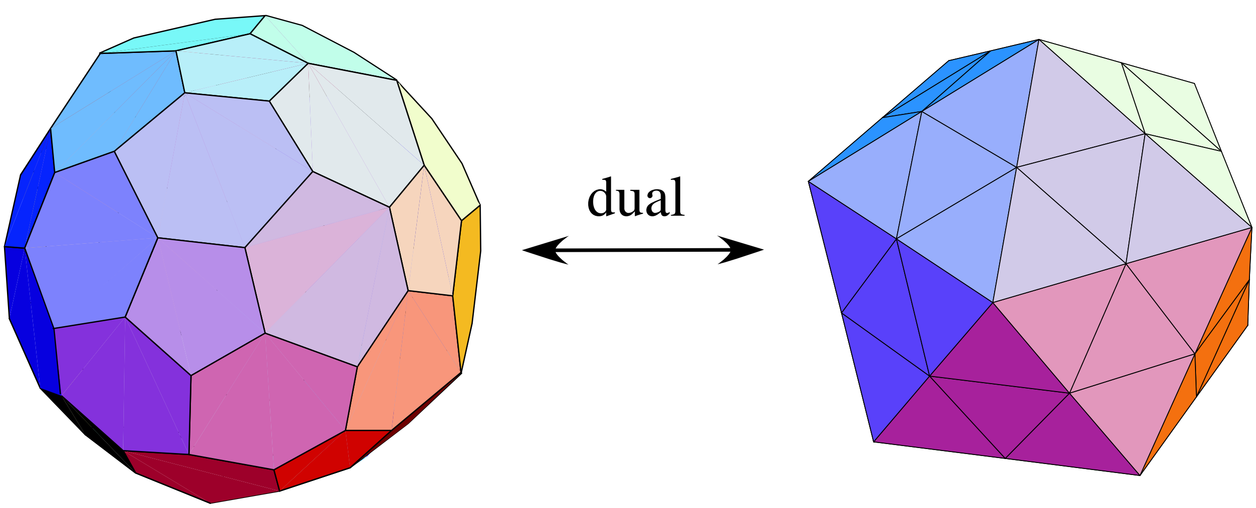
Definition: A Buckyball is **spherical** if the 12 pentagons are evenly distributed around the Buckyball.







Better Definition: A Buckyball B is called **d-spherical** if there exists a bijection M from the vertices of the icosahedron I to the pentagons of B such that if v_1 and v_2 are adjacent vertices of I then the pentagons $M(v_1)$ and $M(v_2)$ are edge-distance d away from each other.

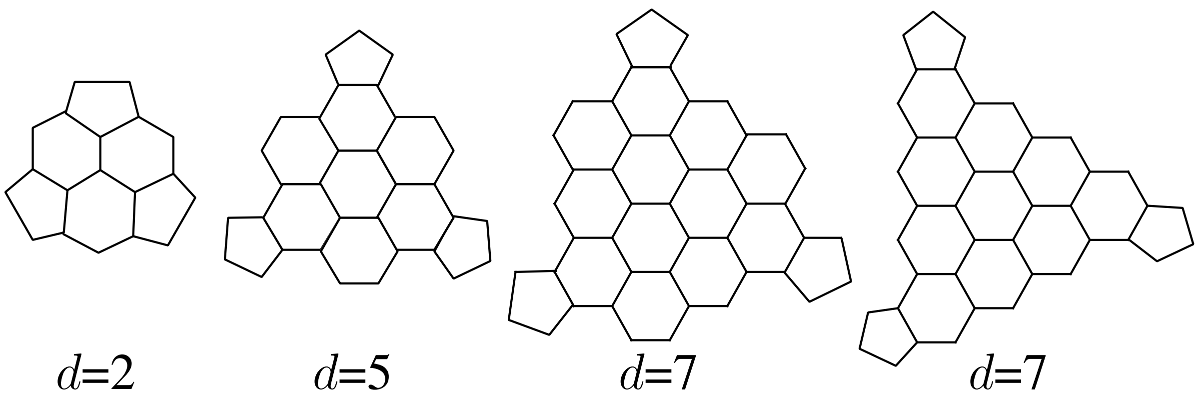


Question: How many different ways can we put hexagons between the pentagons?

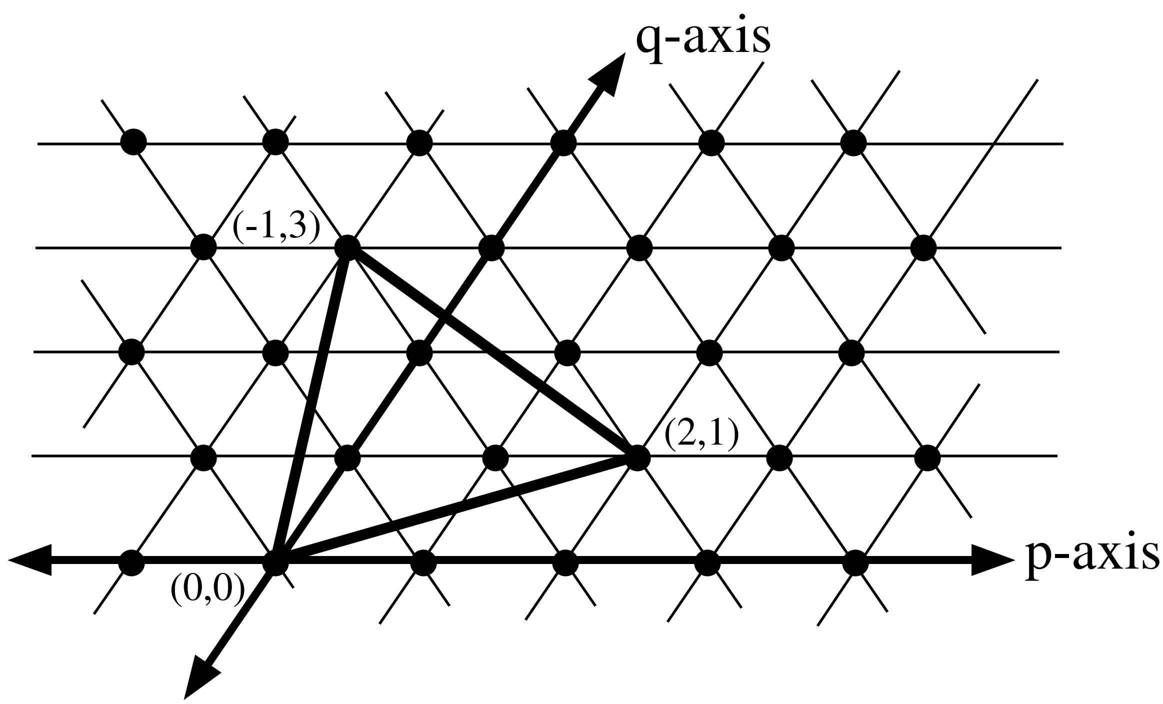
Buckyball Tiles

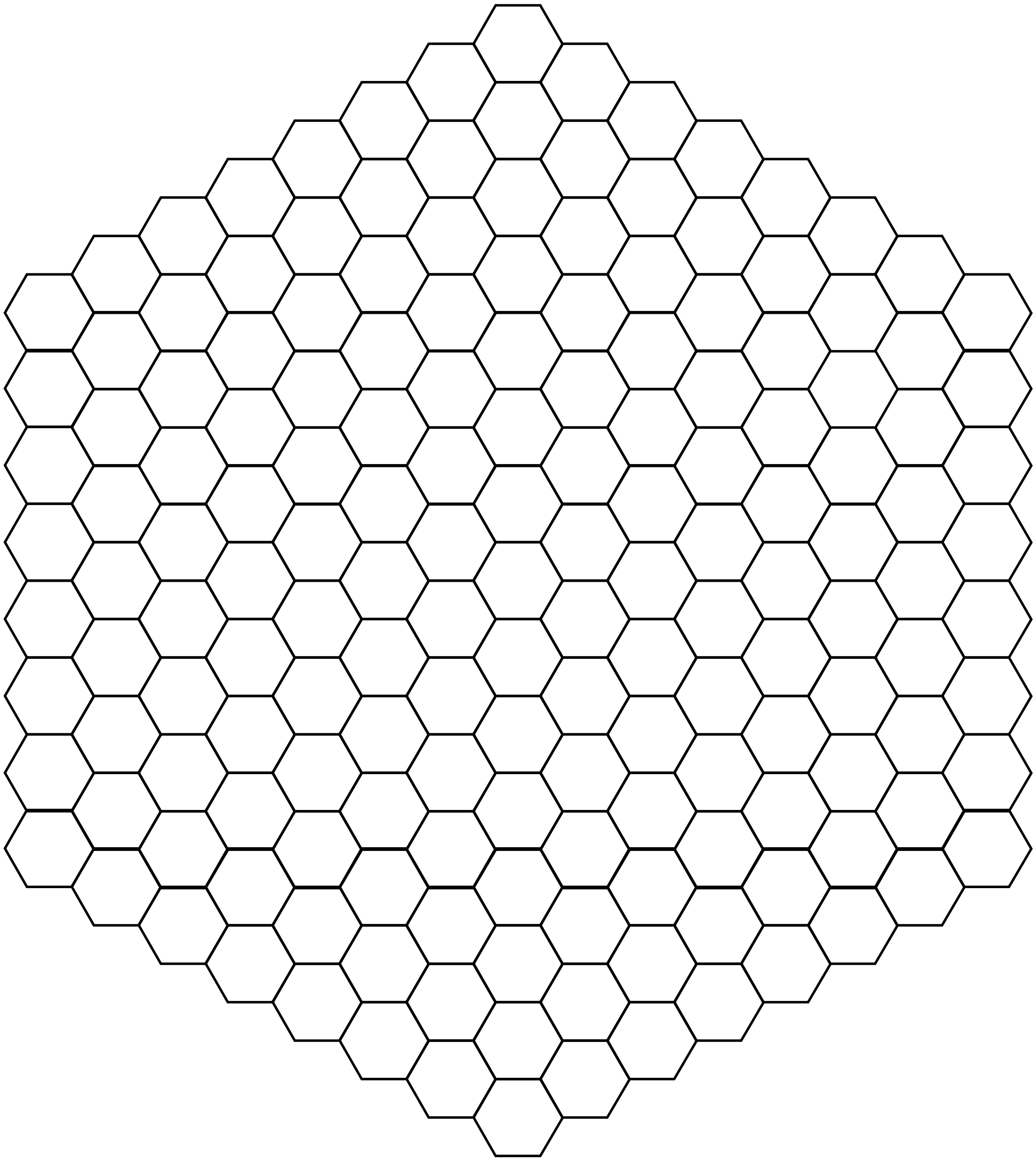
Let v_1, v_2, v_3 be vertices of a face of I.

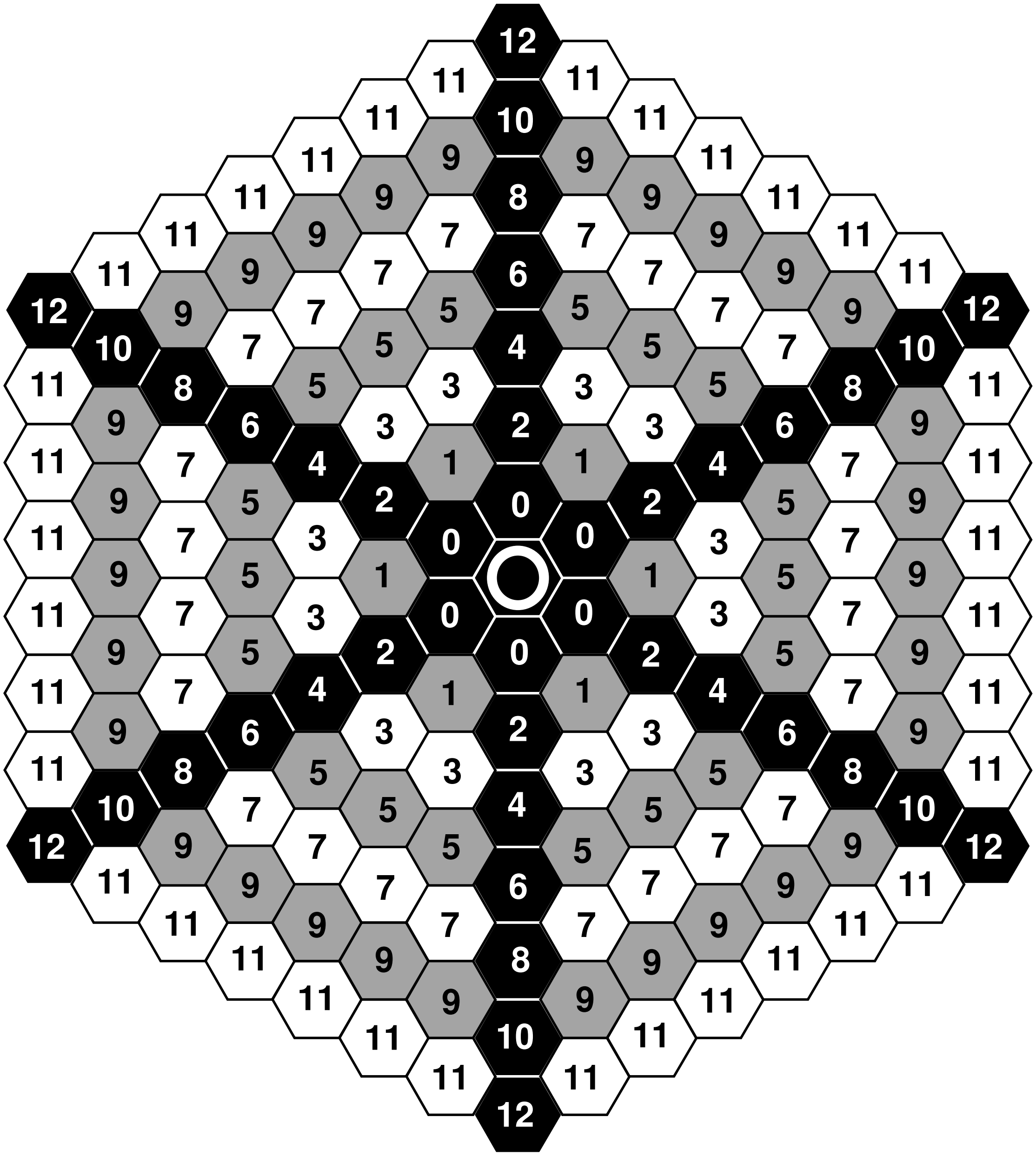
The **tile** of a spherical Buckyball is a set of three pentagons $M(v_1)$, $M(v_2)$, $M(v_3)$ and the hexagons "in between" them.



From the **dual** perspective, a tile is any triangle drawn on the triangle lattice.





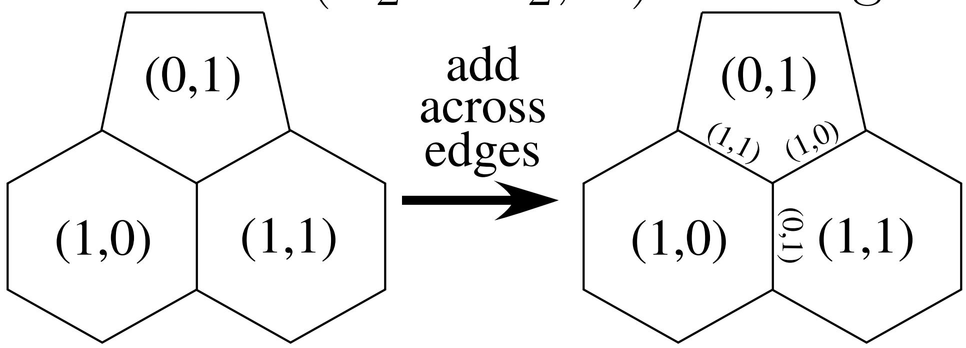


Theorem 1. Let $B(d)$ be the number of different d-spherical Buckyballs, where reflections are considered different. Then

$$B(d) = \begin{cases} 1 & \text{if } d \text{ is even,} \\ (d+1)/2 & \text{if } d \text{ is odd.} \end{cases}$$

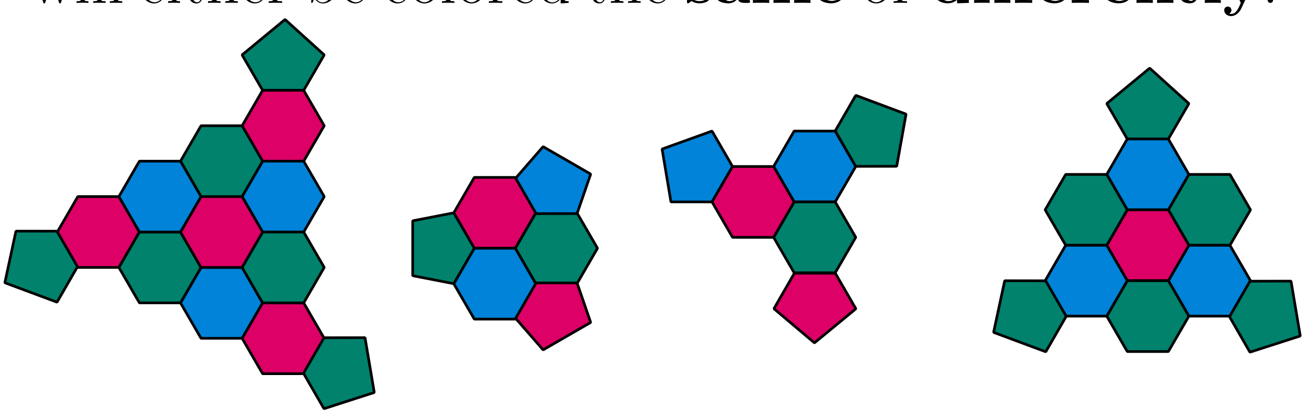
3-edge coloring spherical Buckyballs

Strategy: We'll **4-face color** the Buckyballs. We can then convert this to a 3-edge coloring using the classic $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ coloring.



We'll get our 4-face coloring by coloring the Bucky tiles appropriately.

Given a spherical Buckyball, let H be the tile that generates it. 3-face-color H using the (unique) coloring of the hexagonal lattice. The pentagons will either be colored the **same** or **differently**.

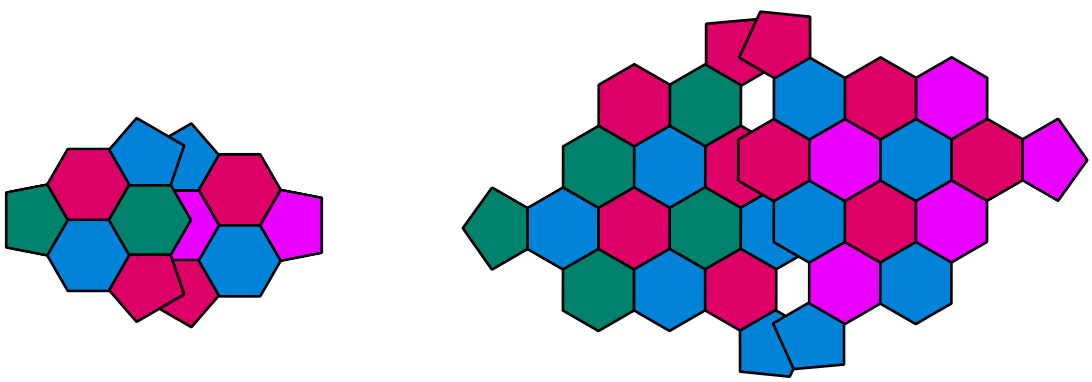


3-edge coloring spherical Buckyballs

Case 1: The pentagons have different colors.

Then start with a 4-vertex coloring of the icosahedron I and let these colors map to the pentagons.

This will determine which three colors to use on each tile. (Overlapping hexagons can be colored either way.)



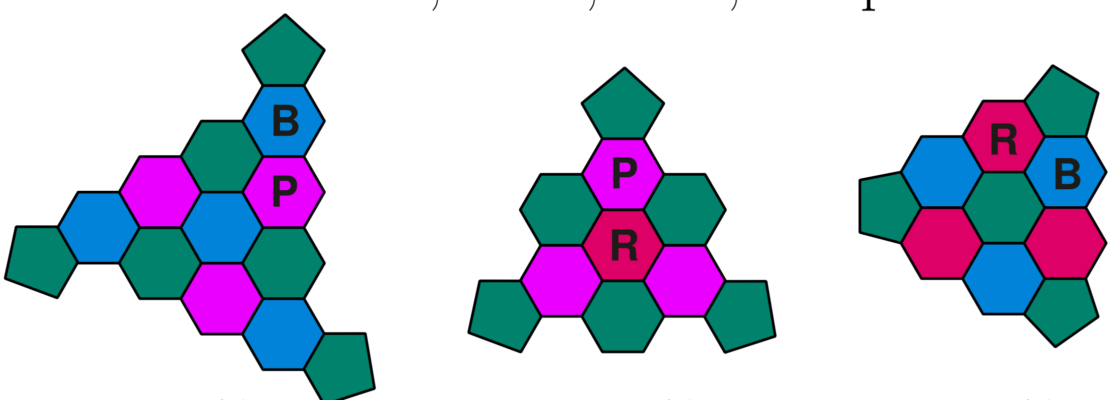
3-edge coloring spherical Buckyballs

Case 2: The pentagons have the same color.

Notation: Suppose this color is **Green**. Pick two reference hexagons h_1 and h_2 in the tile, each from a different non-green color class.

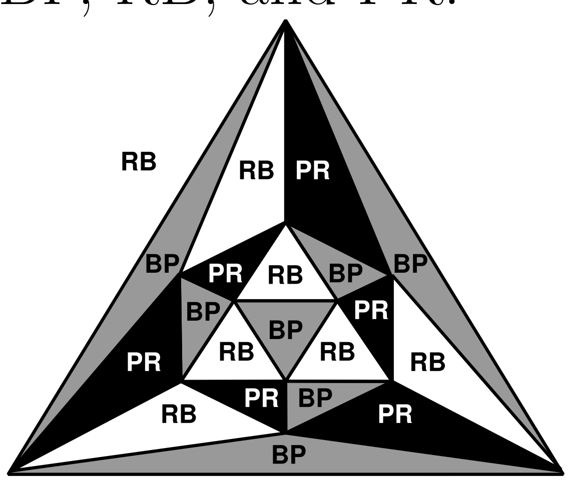
A 3-colored tile will be called XY if h_1 is colored X and h_2 is colored Y.

Use colors Green, Blue, Red, Purple.



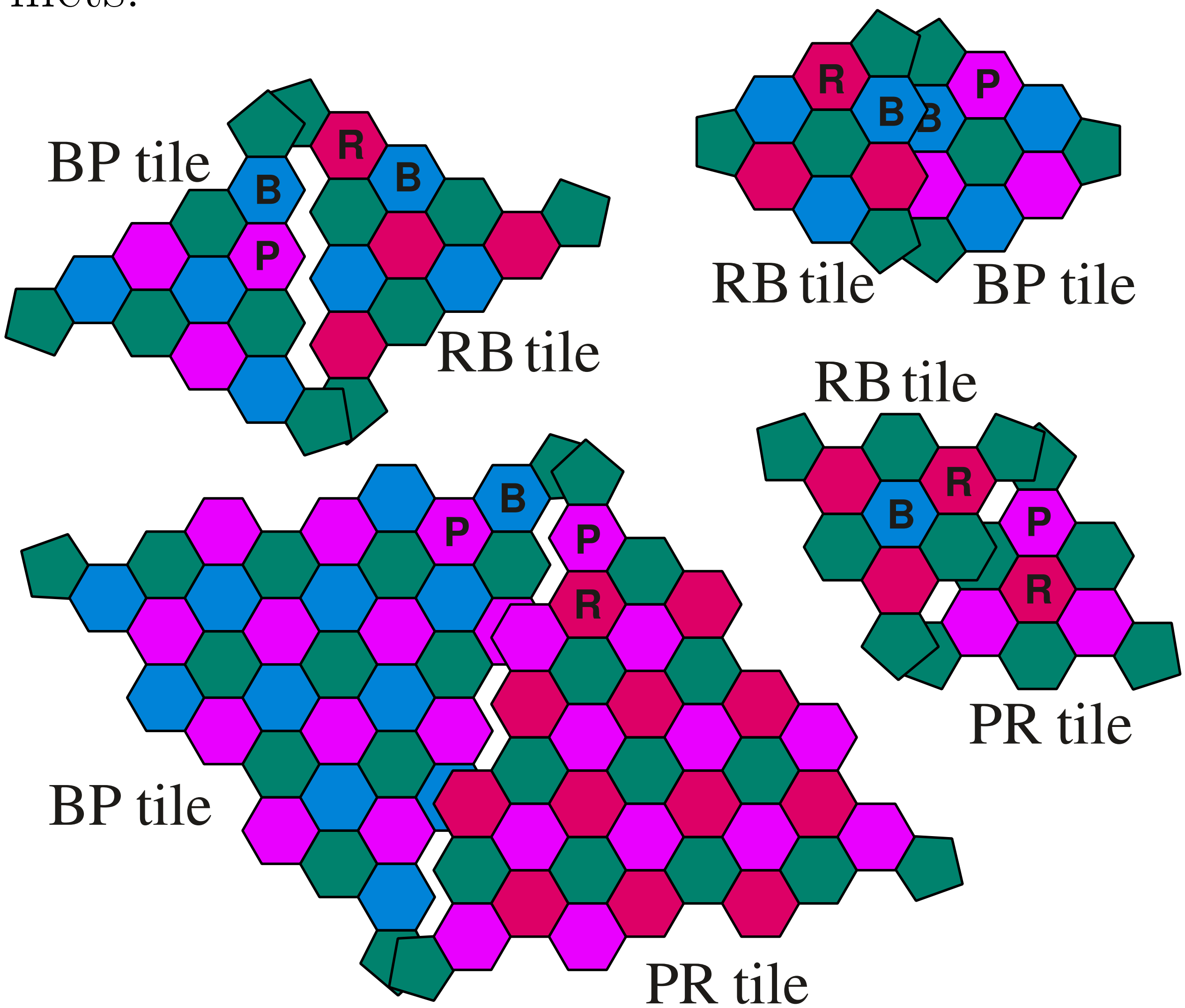
BP tile PR tile RB tile

Now color the faces of the icosahedron with the "colors" BP, RB, and PR.



3-edge coloring spherical Buckyballs

All that's left is to convince yourself that the tile colors BP, RB, and PR won't cause any conflicts.



Since any pair of the colors BP, RB, or PR will share a color, any two adjacent tiles will have this color and Green matching up perfectly. The other two colors will share the same color class.

Final comments

(1) This coloring algorithm is linear in terms of the size of the Buckyball (d or the number of vertices, etc.).

(2) Do the colorings generated by this algorithm generate Hamilton cycles? No, it doesn't. At least not in any of the examples I've done.

A preprint of these results (and/or these talk slides) will eventually appear on my web page:

`http://www.merrimack.edu/~thull/papers/papers.html`