Configuration Spaces for Flat Vertex Folds

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1 Introduction

Flat vertex folds are crease patterns with a single vertex that lie in the plane when collapsed. Two well-known results about flat vertex folds are the Kawasaki-Justin Theorem, which states that a vertex will fold flat if and only if the sum of every other angle between the creases equals π, and the Maekawa-Justin Theorem, which states that the difference between the number of mountain and valley creases must always be two at a flat vertex fold. (See [3], [4], and [1] for details and other results.)

In this paper, we focus on more combinatorial issues. Given a single-vertex crease pattern v with specified crease angles but no mountain-valley (MV) assignment, we may count the number of possible valid (physically realizable) MV assignments. This total, denoted $C(v)$, can be determined in linear time ([1], [2]). If we know only the number of creases, say $2n$, but not the crease angles, we can still obtain sharp bounds on $C(v)$:

$$2^n \leq C(v) \leq 2\binom{2n}{n-1}. \tag{1}$$

We know that $C(v)$ is always even, as the MV parity of the creases can always be flipped. But can $C(v)$ achieve all even values between the bounds in (1)? The answer turns out to be, "No," which immediately makes us wonder whether we could predict or classify the various values of $C(v)$.

We will approach this question by describing a *configuration space* for flat vertex folds. A configuration space is, typically, a geometric object that is used to visualize the range of possibilities for physical or mathematical situation. This is done by quantifying the essential variables of the situation and letting these be parameters along different coordinate axes; the combination

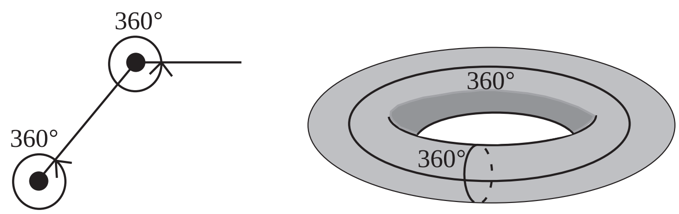

Figure 1: The configuration space for a 2-joint robot arm is a torus.

of these forms the configuration space. One classic example is studying the range of movement for a robot arm with two joints, as illustrated in Figure 1. If each joint has a planar 360° range of rotation, then each joint can be a variable ranging from 0 to 2π, and thus the configuration space is the square $[0, 2\pi] \times [0, 2\pi]$. However, the points 0 and 2π should be identified for each variable, which makes the square "wrap around" and form a torus. Each point on the surface of this torus represents a specific configuration of the robot arm.

Our goal in this paper is to describe the configuration space for a flat vertex fold of degree $2n$.

2 The degree 4 case

We begin by examining the $n = 2$ case in which our flat vertex fold has four crease lines. Let $\alpha_1, ..., \alpha_4$ be the angles, in order, between the creases of our vertex v. The Kawasaki-Justin Theorem tells us that $\alpha_3 = \pi - \alpha_1$ and $\alpha_4 = \pi - \alpha_2$. All four angles are determined by α_1 and α_2, so α_1 and α_2 can be the parameters of our configuration space.

Assign α_1 to our first coordinate and α_2 to our second coordinate. Notice that the range for these parameters is $0 < \alpha_1, \alpha_2 < \pi$, since if either were zero, we wouldn't have four creases, and if either were π then one of α_3, α_4 would be zero. Furthermore, if we pick any α_1 and α_2 between 0 and π, we can let $\alpha_3 = \pi - \alpha_1$ and $\alpha_4 = \pi - \alpha_2$ to obtain angles for a degree four flat vertex fold, showing that (α_1, α_2) must be in our configuration space. Therefore the configuration space for degree four flat vertex folds, which we'll denote P_4, is the open square (see Figure 2)

$$P_4 = (0, \pi) \times (0, \pi).$$

Now, within P_4 there exist subsets for the different values of $C(v)$. The

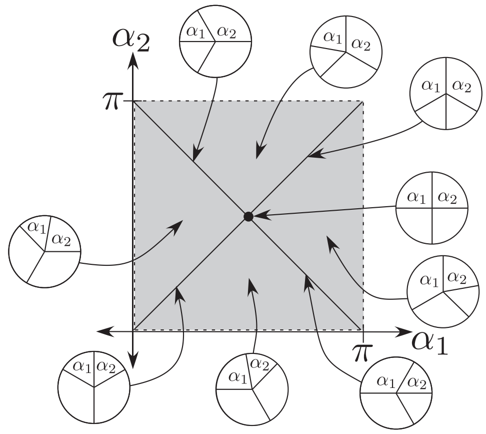

Figure 2: Decomposing P_4 into $C(v)$ subsets.

maximal $C(v)$ is 8, which corresponds to all the angles being equal. This is the point $(\pi/2, \pi/2)$ in P_4.

Next is $C(v) = 6$, and this occurs when two adjacent angles are equal and different from the other pair. For example, we could have $\alpha_1 = \alpha_2$ (which implies that $\alpha_3 = \alpha_4$). This corresponds to the line $y = x$ in P_4, for $0 < x < \pi/2$ and $\pi/2 < x < \pi$. Or we could have $\alpha_2 = \alpha_3$, which implies that $\alpha_2 = \pi - \alpha_1$ (which forces $\alpha_1 = \alpha_4$) and gives us the line $y = \pi - x$ in P_4 for $0 < x < \pi/2$ and $\pi/2 < x < \pi$.

The remaining regions of P_4 are open right triangles, and these correspond to $C(v) = 4$ cases. For example, the region bounded by the y-axis, $y = x$ and $y = \pi - x$ has $\alpha_1 < \alpha_2$, $\alpha_1 < \pi/2$, and $\alpha_2 < \pi - \alpha_1$. Kawasaki-Justin then gives us that $\alpha_3 > \pi/2$ and $\alpha_1 < \alpha_4$. In other words, α_1 is the unique smallest angle. Therefore, the creases surrounding α_1 must have different MV parity. (If they were the same, then in the folded model we would have two large angles covering a smaller angle on the same side of the paper, forcing a self-intersection.) Thus there are two ways to assign Ms and Vs to those creases, and then the others must either both be M or both be V to satisfy the Maekawa-Justin Theorem, yielding $C(v) = 4$. The analysis is similar for the other three triangular regions.

This decomposition of P_4 into subsets gives us a complete classifications of all the possibilities for $C(v)$. This is summarized in Figure 2.

3 Higher dimensions

The configuration spaces P_{2n} quickly become very difficult to visualize for $n > 2$, as they are bounded, open sets in $\mathbb{R}^{2n-2}$.

Example 1. Consider $n = 3$. Letting $\alpha_1, ..., \alpha_6$ be the angles, we can express α_5 and α_6 in terms of the other angles (using Kawasaki-Justin), and thus we may parameterize P_6 by the angles $\alpha_1, ..., \alpha_4$. That is, $P_6 \subset \mathbb{R}^4$. Our reasoning from the $n = 2$ case as well as the Kawasaki-Justin conditions $\alpha_1 + \alpha_3 + \alpha_5 = \alpha_2 + \alpha_4 + \alpha_6 = \pi$ give us the following restrictions on the angles:

$$0 < \alpha_i < \pi \text{ for all } i,\ 0 < \alpha_1 + \alpha_3 < \pi, \text{ and } 0 < \alpha_2 + \alpha_4 < \pi. \qquad (2)$$

This means that the 2-dimensional cross-section of P_6 along the $\alpha_1\alpha_2$-coordinate plane will be an open square, as in the $n = 2$ case. However, the 2-dimensional cross-section along the $\alpha_1\alpha_3$-plane will be an open triangle bounded by $\alpha_1 > 0$, $\alpha_3 > 0$, and $\alpha_3 < \pi - \alpha_1$. See Figure 3.

In fact, any point $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ satisfying (2) will be part of a viable degree 6 flat vertex fold (along with the proper angles α_5 and α_6 given by Kawasaki-Justin) and thus be in P_6. That is, P_6 is an open set. The closure of this set, $\overline{P_6}$, will have as extreme points (vertices) all angle configurations that give equality for the equations (2) and that are the most degenerate, where one of the angles $\alpha_1, \alpha_3, \alpha_5$ equals π, one of the angles $\alpha_2, \alpha_4, \alpha_6$ equals π, and the rest are zero. Thus $\overline{P_6}$ is the polytope formed by the convex hull of the points

$$(0,0,0,0), (\pi,0,0,0), (0,\pi,0,0), (0,0,\pi,0), (0,0,0,\pi)$$

$$(\pi,\pi,0,0), (\pi,0,0,\pi), (0,\pi,\pi,0), (0,0,\pi,\pi).$$

(This can also be seen by viewing the inequalities in (2) as defining the supporting hyperplanes for the polytope $\overline{P_6}$.)

The bounds from (1) give us that $8 \leq C(v) \leq 30$. Examining all the possible cases for six angles around a vertex (which is doable, if somewhat arduous) and using the recursive equations in [2] shows that we have

$$C(v) \in \{8, 12, 16, 18, 20, 24, 30\}.$$

Thus we see that $C(v)$ does not take on all possible values between the bounds in (1). Nonetheless, each of these values should correspond to a subset of P_6.

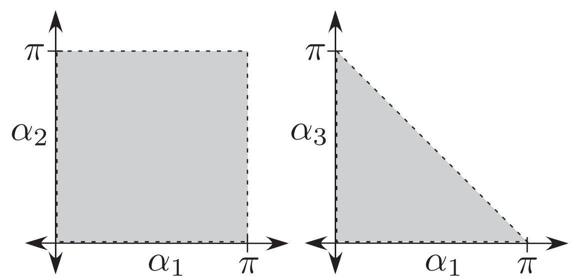

Figure 3: $\alpha_1\alpha_2$-plane and $\alpha_1\alpha_3$-plane slices of P_6.

Rather than focusing on small cases, let us say what we can about the arbitrary-dimension case and then return to P_6.

Let P_{2n} be the configuration space for flat vertex folds of degree $2n$. If our angles are, in order $\alpha_1, ..., \alpha_{2n}$, we know by Kawasaki-Justin that the space can be parameterized by $\alpha_1, ..., \alpha_{2n-2}$. In other words, $P_{2n} \subset \mathbb{R}^{2n-2}$.

We say that a point $x = (\alpha_1, ..., \alpha_{2n-2}) \in \mathbb{R}^{2n-2}$, where $\alpha_i \geq 0$, *corresponds to a set of angles* if there exists $\alpha_{2n-1}, \alpha_{2n} \geq 0$ such that $(\alpha_1, ..., \alpha_{2n})$ satisfy the Kawasaki-Justin conditions. (That is, if $\alpha_{2n-1} = \pi - (\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-3})$ and $\alpha_{2n} = \pi - (\alpha_2 + \alpha_4 + \cdots + \alpha_{2n-2})$.) Note that this corresponding set of angles might not be a degree-$2n$ flat vertex fold, since the definition allows some of the angles to be zero or π.

Theorem 1. *P_{2n} is an open set. Furthermore, if $x \in \overline{P_{2n}} - P_{2n}$ (the boundary of P_{2n}), then x corresponds to a degenerate set of angles where at least one of the angles α_i equals 0 or π.*

Proof. The fact that all angles in a degree-$2n$ flat vertex fold must be nonzero and less that π, together with the Kawasaki-Justin conditions, give us that every point in P_{2n} must satisfy the inequalities

$$0 < \alpha_i < \pi \ \text{ for all } i, \ 0 < \alpha_1 + \alpha_3 + \cdots + \alpha_{2n-3} < \pi, \\ \text{and } 0 < \alpha_2 + \alpha_4 + \cdots + \alpha_{2n-2} < \pi. \tag{3}$$

Furthermore, any point satisfying these equations must be in P_{2n}, which proves that P_{2n} is open. Any point x on the boundary of P_{2n} must also satisfy the equations (3) but have at least one of the inequalities being an equality. Thus x corresponds to a set of angles $\alpha_1, ..., \alpha_{2n}$ where either at least one of the α_i is 0 or π for some $1 \leq i \leq 2n-2$ (in which case, we're done) or $\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-3}$ equals 0 or π or $\alpha_2 + \alpha_4 + \cdots + \alpha_{2n-2}$ equals

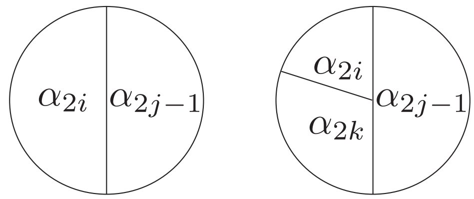

Figure 4: Degenerate angle configurations for a vertex (left) and an edge (right) of $\overline{P_{2n}}$.

0 or π. These latter two cases imply that either α_{2n-1} or α_{2n} equals 0 or π. Thus every case results in x corresponding to a set of angles where at least one of the α_i equals 0 or π. □

We can use Theorem 1 to examine more carefully the faces of $\overline{P_{2n}}$. The vertices of $\overline{P_{2n}}$, for example, will correspond to the most extreme degenerate degree-$2n$ flat vertex folds, where two angles are equal to π and the rest are equal to zero. In order for such a case to satisfy Kawasaki-Justin one of the π angles must be an even-indexed angle and the other be an odd-indexed angle. This is illustrated in the left side of Figure 4.

Thus we have that $\overline{P_{2n}}$ has n^2 vertices whose coordinates are $(\alpha_1, ..., \alpha_{2n-2})$ where at most one of the $\alpha_{2i} = \pi$, at most one of the $\alpha_{2i+1} = \pi$, and the remaining $\alpha_i = 0$. (If all the $\alpha_i = 0$ then we have $\alpha_{2n-1} = \alpha_{2n} = \pi$ in the corresponding set of angles.)

An edge (1-face) of $\overline{P_{2n}}$ will be a line segment of points $E(u, v) = \{\lambda u + (1 - \lambda)v : 0 \leq \lambda \leq 1\}$ connecting two vertices u and v where the points of $E(u, v)$, aside from the endpoints, correspond to slightly-less-extreme degenerate degree-$2n$ flat vertex folds than those of the vertices. That is, instead of having an even-indexed angle and an odd-indexed angle equaling π as we did for the vertices, each point in the relative interior of $E(u, v)$ will correspond to a set of angles with either one even-indexed angle equaling π and two odd-indexed angles adding to π, or vice-versa (one odd-indexed angle is π and two even-indexed angles sum to π). All the other angles would have to be zero; see the right side of Figure 4. Thus, if the non-zero corresponding set of angles for the vertex u are at coordinate positions $2i$ and $2j - 1$ and those for v are at coordinate positions $2s$ and $2t - 1$, then either $i = s$ or $j = t$ must be true in order for $E(u, v)$ to be an edge of $\overline{P_{2n}}$. That is, u and v must have a π in a common coordinate so that their other π coordinates can switch places as we travel along the edge $E(u, v)$.

The number of edges of $\overline{P_{2n}}$ will therefore be $\binom{n}{1}\binom{n}{2} + \binom{n}{2}\binom{n}{1}$, because in the corresponding set of angles $(\alpha_1, ..., \alpha_{2n})$ we could choose one of the n even-indexed angles to be π, two of the n odd-indexed angles to sum to π, and the rest to be 0, or we could pick two even-indexed angles to sum to π, one of the odd-indexed angles to be π, and the rest to be 0.

The 2-faces of $\overline{P_{2n}}$ follow similarly. In the corresponding set of angles for any point of a 2-face we could have one even-indexed angle $\alpha_{2i} = \pi$ (and the rest $= 0$) and three odd-indexed angles $\alpha_{2j-1}, \alpha_{2k-1}$, and α_{2l-1} being non-zero but adding up to π (and the rest $= 0$). This gives us two parameters (say α_{2j-1} and α_{2k-1}, which then determine α_{2l-1}) and thus will span a 2-face. Or we could chosen have two even-indexed angles and two odd, or three even-indexed angles and one odd. Thus there are $\binom{n}{1}\binom{n}{3} + \binom{n}{2}\binom{n}{2} + \binom{n}{3}\binom{n}{1}$ 2-faces total.

Thus we obtain the following:

Theorem 2. *The number of k-cells in $\overline{P_{2n}}$ is*

$$f_k = \sum_{i=0}^{k} \binom{n}{i+1}\binom{n}{k-i+1} = \binom{2n}{k+2} - 2\binom{n}{k+2}.$$

Proof. The previous arguments illustrate how we obtain the summation, and the summation identity can be obtained via standard combinatorial methods such as generating functions. We also offer a different combinatorial reasoning: To count f_k we want to pick $k+2$ angles from the $2n$ corresponding angles to be non-zero in order to create our degenerate flat vertex fold. But we don't want all of the angles to be even-indexed or all odd-indexed, so we subtract the $2\binom{n}{k+2}$ ways in which this can happen. The result is all the ways to have all angles zero except for $k+2$ of them, where some are even-indexed and some odd-indexed. The even-indexed angles must sum to π, and so must the odd-indexed angles. This means that to parameterize these degenerate cases we don't need all of the $k+2$ angles; we can eliminate one of the even-indexed angles and one of the odd-indexed angles, leaving us with k parameter coordinates for this face, thus creating a k-face. □

The arguments given for Theorem 2 provide everything needed to calculate the coordinates for the vertices, edges, etc. of $\overline{P_{2n}}$, which can then be generated using *Mathematica* or other visualization software.

Figure 5 shows a projection of $\overline{P_6}$. We can try to compare our general calculations with the intuition developed earlier for the degree six flat vertex

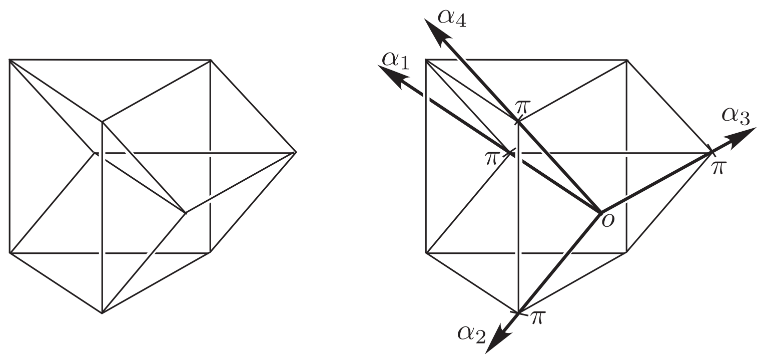

Figure 5: A projection of the 4-D polytope $\overline{P_6}$.

fold case. For example, the right side of Figure 3 shows how slicing P_6 along the $\alpha_1\alpha_3$-plane gives a right triangle. To make such a slice a 2-face of $\overline{P_6}$, we'd need the other angles (the even-indexed ones) to be extreme, either 0 or π, while still obeying Kawasaki-Justin. So we could have

$$\begin{array}{ll} (\alpha_1, 0, \alpha_3, 0) & \text{where } 0 \le \alpha_1 + \alpha_3 \le \pi \text{ and } \alpha_6 = \pi \\ (\alpha_1, \pi, \alpha_3, 0) & \text{where } 0 \le \alpha_1 + \alpha_3 \le \pi \text{ and } \alpha_6 = 0 \\ (\alpha_1, 0, \alpha_3, \pi) & \text{where } 0 \le \alpha_1 + \alpha_3 \le \pi \text{ and } \alpha_6 = 0. \end{array}$$

The same reasoning applies to slices along the $\alpha_2\alpha_4$-plane, giving $\overline{P_6}$ six faces which will be 45° right triangles. Careful examination of Figure 5 reveals these faces.

In fact, going back to the general case, we can be more specific about the structure of $\overline{P_{2n}}$. Let $e_i \in \mathbb{R}^{2n-2}$ be the point with 0 for every coordinate except the ith, which is π. Let o denote the origin. We denote the convex hull of a finite set of points x_i by $\text{conv}(x_1, ...x_n) = \{\lambda_1 x_1 + \cdots + \lambda_n x_n : \lambda_i \ge 0, \sum_{i=1}^n \lambda_i = 1\}$. Define

$$EP_{2n} = \text{conv}(o, e_2, e_4, ..., e_{2n-2}) \text{ and } OP_{2n} = \text{conv}(o, e_1, e_3, ..., e_{2n-3}).$$

Then EP_{2n} and OP_{2n} are both $(n-1)$-simplices in $\mathbb{R}^{2n-2}$.

Lemma 1. *$x \in EP_{2n}$ (resp. OP_{2n}) if and only if $x = \lambda_1 e_2 + \lambda_2 e_4 + \cdots + \lambda_{n-1} e_{2n-2}$ (resp. $x = \lambda_1 e_1 + \lambda_2 e_3 + \cdots + \lambda_{n-1} e_{2n-3}$) where $\lambda_i \ge 0$ and $\sum_{i=1}^{n-1} \lambda_i \le 1$.*

Proof. If $x \in EP_{2n}$ or OP_{2n} then certainly x can be written as described in the lemma, since $\lambda_0 o$ is just the zero vector. For the other direction, if we write $x = \lambda_0 o + \lambda_1 e_2 + \cdots + \lambda_{n-1} e_{2n-2}$ where $\lambda_0 = 1 - \sum_{i=1}^{n-1} \lambda_i$ then we have that $x \in EP_{2n}$. The same argument with the points e_{2i} switched to e_{2i-1} handles the OP_{2n} case. $\square$

Recall that if A and B are sets of points then their *Minkowski sum* is $A + B = \{x + y : x \in A, y \in B\}$.

Theorem 3. $\overline{P_{2n}} = OP_{2n} + EP_{2n}$.

Proof. Note that $x = (\alpha_1, ..., \alpha_{2n-2}) \in \overline{P_{2n}}$ if and only if

$$\begin{aligned} x &= \frac{\alpha_1}{\pi} e_1 + \frac{\alpha_2}{\pi} e_2 + \cdots + \frac{\alpha_{2n-2}}{\pi} e_{2n-2} \\ &= (\frac{\alpha_1}{\pi} e_1 + \frac{\alpha_3}{\pi} e_3 + \cdots + \frac{\alpha_{2n-3}}{\pi} e_{2n-3}) + (\frac{\alpha_2}{\pi} e_2 + \frac{\alpha_4}{\pi} e_4 + \cdots + \frac{\alpha_{2n-2}}{\pi} e_{2n-2}) \end{aligned}$$

where $0 \le \alpha_i \le \pi$ for all i and the coordinates of x correspond to a set of angles that satisfy the Kawasaki-Justin Theorem. These conditions on x are satisfied if and only if $0 \le \alpha_i/\pi \le 1$, $\sum_{i=1}^{n-1} \alpha_{2i-1} \le \pi$ and $\sum_{i=1}^{n-1} \alpha_{2i} \le \pi$, i.e.,

$$\frac{\alpha_1}{\pi} + \frac{\alpha_3}{\pi} + \cdots \frac{\alpha_{2n-3}}{\pi} \le \frac{\pi}{\pi} = 1 \text{ and } \frac{\alpha_2}{\pi} + \frac{\alpha_4}{\pi} + \cdots \frac{\alpha_{2n-2}}{\pi} \le \frac{\pi}{\pi} = 1.$$

Thus by Lemma 1 we have that $x \in \overline{P_{2n}}$ if and only if $x \in OP_{2n} + EP_{2n}$. $\square$

In other words, $\overline{P_{2n}}$ is the sum of two $(n-1)$-simplices.

4 Generalizations and future work

Flat vertex folds do not need to be restricted to geometrically flat paper. If we place the vertex of our fold at the tip of a cone-shaped piece of paper, then we can consider folding it up. As described in [1] and [2], the Kawasaki-Justin and Maekawa-Justin Theorems still hold (with some modifications) for folding cones. For example, instead of $0 < \alpha_i < \pi$ for each angle, we have, if ρ is the cone angle of the cone, that $0 < \alpha_i < \rho/2$. Also, the Kawasaki-Justin conditions become: the sum of every other angle around the vertex $= \rho/2$.

Therefore we could extend the configuration space P_{2n} by adding an axis to parameterize the cone angle of the paper. Because changing the cone angle restricts the angles α_i, this effectively turns our configuration space into an infinite cone. The case $n = 2$ is illustrated in Figure 6.

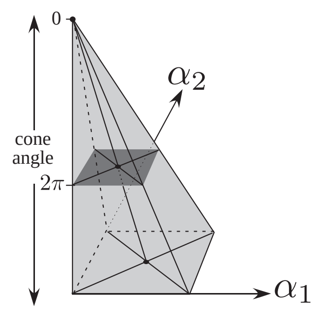

Figure 6: P_4 extended to include the cone angle.

There is one caveat to this cone angle generalization: if the cone angle is $> 2\pi$ then a different kind of flat folding can be done, one where the vertex is neither convex nor concave but flat, with the excess paper layered radially around it. The configuration space described here does not include such cases.

Additional work needs to be done to see whether or not our descriptions of the spaces P_{2n} can determine what values $C(v)$ can attain and the subsets to which they correspond.

Acknowledgments. The author would like to thank sarah-marie belcastro and Günter Rote for valuable comments and corrections.

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