Defective List Colorings of Planar Graphs

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Abstract

We combine the concepts of list colorings of graphs with the concept of defective colorings of graphs and introduce the concept of defective list colorings. We apply these concepts to vertex colorings of various classes of planar graphs.

A *defective coloring* with defect d is a coloring of the vertices such that each color class corresponds to an induced subgraph with maximum degree at most d. A *k-list assignment* L, is an assignment of sets to the vertices so that $|L(v)| = k$, for all vertices v and an *L-list coloring* is a coloring such that the color assigned to v is in $L(v)$ for all vertices v, and a *d-defective L-list coloring* is an L-list defective coloring with defect d.

For a given graph G and defect d, we are interested in the smallest number k such that any k-list assignment, L, is d-defective L-list colorable. We will show that for outerplanar graphs, any 2-list assignment, L, has a 2-defective L-list coloring, and that this is best possible. We give results of this form pertaining to triangle-free outerplanar graphs and bipartite planar graphs. In general, we prove that all planar graphs are 2-defective L-list colorable for any 3-list assignment L.

1 Introduction

The subject of list colorings was introduced independently by Erdös, Rubin and Taylor ([3]) and Vizing ([8]) during the 1970's and has received considerable attention since then. Recently, some exciting research has been done on list colorings of planar graphs. A *k-list assignment* $\{L(v) : v \in V(G)\}$, is an assignment of sets to the vertices so that $|L(v)| = k$, for all vertices v. An *L-list coloring* is a coloring such that the color assigned to v is in $L(v)$

for all vertices v. If the coloring is proper, we say that G is *L-list colorable*. Also, G is *k-choosable* (or *k-list colorable*) if given any k-list assignment, L, G is L-list colorable. In 1994, a short elegant proof that all planar graphs are 5-choosable was discovered by Thomassen, see [7]. This answered a 1979 conjecture of Erdös, Rubin and Taylor ([3]). Further, two examples of planar graphs that are not 4-choosable have been published. One by M. Mirzakhani in 1996 ([5]), and the other by M. Voigt in 1993 ([9]). As for other classes of planar graphs, it is easy to see that outerplanar and triangle free outer planar graphs are 3-choosable and that is the best one can say due to odd cycles. Alon and Tarsi proved in [1] that all bipartite planar graphs are 3-choosable. The proof of this is nontrivial and involves an algebraic method.

Cowen, Cowen and Woodall consider *defective* colorings of graphs. A graph G is said to be *k-colorable with defect d*, or simply *(k, d)-colorable*, if the vertices of G can be colored with k colors in such a way that each vertex is adjacent to at most d vertices of the same color as itself. In [2], Cowen, Cowen and Woodall give the following results pertaining to planar graphs: All planar graphs are (3,2)-colorable and there exists one which is not (3,1)-colorable; outerplanar graphs are (2,2)-colorable and there exists one which is not (2,1)-colorable. Their arguments are seen to be particularly concise, especially when one considers the defect $d = 0$ case. For a definition of *outerplanar*, see Section 2.

In this paper we combine these two concepts in a natural way and introduce *defective list colorings*. A *d-defective L-list coloring* is an L-list defective coloring with defect d. If a graph is d-defective L-list colorable for all k-list assignments L then we say G is *(k, d)-choosable*. For a given graph G and defect d, we are interested in the smallest number k such that G is (k, d)-choosable.

In this paper, we present results about different classes of planar graphs. We prove that all planar graphs are $(3, 2)$-choosable. This is best possible for lists of size 3 since there exists a planar graph that is not $(3, 1)$-colorable, see [2], and as a result, this happens to imply Cowen, Cowen and Woodall's result that all planar graphs are $(3, 2)$-colorable. It is unknown whether the best d such that all planar graphs are $(4, d)$-choosable is 1 or 2. We show that all outerplanar graphs are (2,2)-choosable as a corollary to the result about planar graphs in general. This is the best possible, considering the result stated above that there exists an outerplanar graph which is not (2,1)-colorable. This again implies the result about defective colorability that outerplanar graphs are (2,2)-colorable. Also, we show that triangle-free outerplanar graphs are (2,1)-choosable. Since the triangle-free outerplanar graph C_5 provides an example of a graph that is not $(2, 0)$-colorable, this also provides the result that triangle-free outerplanar graphs are (2,1)-colorable. Considering the class of bipartite planar graphs, we give a result which says that the fact that all bipartite planar graphs are $(3, 0)$-choosable is the best possible. Much of this work, in addition to other results on defective list colorings, appears in unabridged form in [4].

2 Notation

For convenience, we include the definitions of some terms and notation. For definitions that are not included here, we ask the reader to refer to a standard text on graph theory, for example, see [11]. Let $N(v)$ denote the neighborhood of vertex v, that is, all vertices adjacent to v. For $A \subseteq V(G)$, $< A >$ indicates the induced graph on A. We use the shorthand $G - v$ to indicate $< V(G)\backslash\{v\} >$ and $G' + v$ to indicate $< V(G') \cup \{v\} >$, whenever G' is a proper, induced subgraph of a graph G and $v \in V(G)$, $v \notin V(G')$.

An *outerplanar graph* is a planar graph that can be embedded in the plane in such a way that all vertices border the unbounded face. The *outer circuit* of a plane graph is the circuit which corresponds to traversing the vertices bordering the unbounded face in a clockwise fashion. Note that the outer circuit, $C = u_1, e_1, u_2, e_2, \ldots, u_k, e_k, u_1$, has vertices and edges, $V(C) = \{v = u_i : i = 1, 2, \ldots, k\}$ and $E(C) = \{e = e_i : i = 1, 2, \ldots, k\}$, respectively. A *chord* of an outer circuit, C, is an edge which has both end vertices in $V(C)$ but is not in $E(C)$. If u and v are vertices of an outer circuit C, then $C(u, v)$ is the path from u to v along C. If G has outer circuit C, $Int(C)$ will denote the induced subgraph on $V(G)\backslash V(C)$.

A plane graph is said to be *nearly triangular* if every bounded face is a triangle.

Given a coloring c and a vertex v of a graph, we say v is colored with *defect* d or $def(v) = d$ if d of its neighbors are colored with the same color as v under the coloring c.

3 Planar Graphs in General

Thomassen's result in [7] gives us that all planar graphs are (5,0)-choosable. If we restrict ourselves to lists of size 4 or 3 we might expect to be able to list color planar graphs with some suitable defect. Determining the smallest defect in both cases has proven to be difficult. We succeeded in showing that all planar graphs are $(3, 2)$-choosable, which is best possible since there exist planar graphs which are not $(3, 1)$-colorable, but leave it open to determine if $d = 2$ is best possible for $(4, d)$-choosability. We note that the examples of non-4-choosable planar graphs in [5] and [9] are 1-defective list colorable for the list assignments given. This situation is illustrated in Table 1. Note that an "O" in the kth row and dth column indicates that planar graphs

		defects				
		0	1	2	3	4
colors	2	X	X	X	X	X
	3	X	X	O	O	O
	4	X	?	O	O	O
	5	O	O	O	O	O

Table 1: Defect choosability of planar graphs

are (k, d)-choosable, and an "X" indicates that a counterexample exists to show the opposite. A "?" indicates areas of uncertainty.

As was done by Thomassen in the proof that all planar graphs are 5-choosable, we prove that all planar graphs are (3,2)-choosable by proving something stronger: if we assign lists of size two to the vertices in the outer circuit of a plane graph G, and lists of size three to the other vertices, then this can be 2-defective list colored. This will give us enough power to make induction work.

Theorem 1 *Let G be a nearly triangular planar graph with outer circuit $C = v_1, e_1, v_2, e_2, \ldots, v_k, e_k, v_1$. Let $\{L(v) : v \in V(G)\}$ be a list assignment with $|L(v)| \geq 2$ for all vertices $v \in V(C)$ and $|L(v)| \geq 3$ otherwise. Let v_1 and v_2 be precolored with colors $c(v_1)$ and $c(v_2)$. Then the coloring c can be extended to a 2-defective L-list coloring of G in such a way that*

- *if $c(v_1) = c(v_2)$ then $def(v_1) = def(v_2) = 1$ and*
- *if $c(v_1) \neq c(v_2)$ then $def(v_2) = 0$ and $def(v_1) \leq 1$.*

Proof: We proceed by induction on the number of vertices in G. The base case is K_3, for which the theorem is obviously true. So let G be a nearly triangular planar graph of order $n > 3$ and assume the Theorem is true for all such graphs on $n - 1$ vertices or less.

Case 1: The graph G has a cut vertex, v_j. Let G' be the component of $G - v_j$ such that $G' + v_j$ contains both v_1 and v_2. Let $G_1 = G' + v_j$ and $G_2 =< V(G)\backslash V(G') >$. We apply the induction hypothesis on G_1. This gives us a color, $c(v_j)$, assigned to v_j. Note that $def_{G_1}(v_j) \leq 2$.

Let $v_i \in C$ be a neighbor of v_j in G_2 and set $c(v_i) \in L(v_i) \setminus \{c(v_j)\}$. Then applying the inductive hypothesis to G_2 will give us a 2-defective coloring of G_2 with $def_{G_2}(v_j) = 0$. Putting the 2 colorings together gives the desired coloring for G.

For the remaining cases we assume that G is a block and hence, C is a cycle. Let $a \in L(v_3) - c(v_2)$. If $c(v_1) = a$, then let $A = \{v \in V(C) : a \in L(v)\}$. Otherwise, let $A = \{v \in V(C) : a \in L(v)\} \setminus \{v_1\}$.

Case 2: The subgraph $< V(C) >$ has no chord v_iv_j such that $v_i, v_j \in A$.

Let $I = Int(C)$ and for all $v \in A$, let $I(v) = N(v) \cap I$ and set $I(A) = \bigcup_{v \in A} I(v)$.

Let $G' =< V(G) \setminus A >$ and create a new list assignment L' for G' as follows: for $v \in I(A)$ let $L'(v) = L(v) \setminus \{a\}$, and otherwise let $L'(v) = L(v)$. Then the outer face, C', of G' has vertices $(C \setminus A) \cup I(A)$. If $v_1 \notin V(C')$, select a neighbor v' immediately preceding v_2 along C' and precolor v' with a color in $L'(v') \setminus \{c(v_2)\}$. We apply the induction hypothesis to G', and use the resulting 2-defective list coloring c' of G' to obtain the coloring c of the vertices of G, with the addition of $c(v) = a$ for all $v \in A$.

We claim that this list coloring satisfies the theorem. Indeed, we have that $def(v_2) = 0$. If $c(v_1) = a$, then at most 1 neighbor of v_1 has the same color as v_1, namely, v_k, and if $c(v_1) \neq a$ then by induction at most one neighbor of v_1 has the same color as v_1.

If $v \in A$ then none of v's neighbors are colored a, except perhaps those in A. But since C is a cycle and $< V(C) >$ has no chord with both endpoints in A we know that $def(v) \leq 2$.

Now consider vertices $v \in V(G') \setminus \{v_1, v_2\}$. If $c'(v) \neq a$, then $\mathrm{def}_G(v) = def_{G'}(v) \leq 2$. If $c(v) = a$, then $v \notin C$, so $v \in I$ and v must not be adjacent to any vertex in A, for otherwise it couldn't have been colored with a. Thus $def_G(v) = def_{G'}(v) \leq 2$.

Case 3: The subgraph $< V(C) >$ has a chord v_iv_j such that $v_i, v_j \in A$.

Let C' be the largest cycle such that $V(C') \subseteq V(C)$, $< V(C') >$ contains v_1 and v_2, and it has no chords v_iv_j with $v_i, v_j \in A$. Let G' be the subgraph with outer circuit C'.

We apply Case 2 to the subgraph G' of G.

Each component $K_{i,j}$ of $G \setminus G'$ is attached to G' by a chord v_iv_j of $< V(C) >$. For each component $K_{i,j}$, set $G_{i,j} =< V(K_{i,j}) \cup \{v_i, v_j\} >$.

In the coloring of G', v_i and v_j have the same color, a. Then for each pair i, j using the two vertices, v_i and v_j as the precolored vertices, we apply the induction hypothesis to the subgraph $G_{i,j}$ of G. Combining all of these colorings, we obtain a coloring c of G.

We check to be sure that this coloring satisfies the theorem. If $c(v_1) = c(v_2)$ then $v_1 \notin A$ and $def(v_1) = def(v_1) = 1$ as in Case 2. If $c(v_1) \neq c(v_2)$ then it is possible that $v_1 \in A$. If not, both v_1 and v_2 got 0 defects as in Case 2. If $v_1 \in A$, then v_2 got 0 defects when applying Case 2 to G' and v_1 got at most 1 defect when coloring G' and no more defects when applying induction to the subgraphs $G_{i,j}$. As for any vertex v_i on a chord v_iv_j, v_i got at most 1 defect when applying Case 2 to G' and no more for each $G_{i,j}$ that it is in.

Thus we have the desired L-list coloring for G. □

Outerplanar

	defects			
colors	0	1	2	3
1	X	X	X	X
2	X	X	O	O
3	O	O	O	O

Outerplanar, Δ-free

	defects			
colors	0	1	2	3
1	X	X	X	X
2	X	O	O	O
3	O	O	O	O

Table 2: Defect choosability of outerplanar graphs

4 Outerplanar Graphs

Outerplanar graphs provide an interesting example for defective colorings. We classify the defective list coloring behavior, for both outerplanar graphs and triangle-free outerplanar graphs, in Table 2.

Lemma 1 *Let G be a triangle-free outerplanar graph with no vertices of degree 1. Then G has two adjacent vertices of degree two.*

Proof: If any end-block of G has no chords then the lemma is obviously true.

Let C be the outer circuit of an end-block of G and let $x \in V(C)$ be a cut vertex of G. Consider all chords, $u'v'$, of C and the shortest path from u' to v' in C, denoted $C(u',v')$. Among all such chords, $u'v'$ such that $x \notin C(u',v')$, let uv be such that the length of $C(u,v)$ is minimal. Let $C(u,v) = \{u, a_1, a_2, ..., a_{\ell-1}, v\}$ be of length ℓ. Let $A = \{a_1, ..., a_{\ell-1}\}$. Then

(i) $\forall a \in A$, a is not part of a chord in C, otherwise we would have a contradiction with minimality of ℓ,

(ii) $\ell - 1 \geq 2$, for otherwise we would have that $\{u, a_1, v\}$ is a triangle, and

(iii) no vertex in A is a cut vertex, otherwise $< C >$ would not be a block.

Thus all vertices in A are of degree 2 and there are at least 2 of them, which completes the proof. □

Note that one could improve this lemma to obtain two pairs of adjacent vertices of degree two in any triangle-free outerplanar graph, but the above will suffice for our purposes.

Theorem 2 *Every triangle-free outerplanar graph is (2,1)-choosable.*

Proof: We proceed by induction on the number of vertices. Clearly any triangle-free outerplanar graph on 4 or fewer vertices is (2,1)-choosable.

So let G be outerplanar and triangle-free on at least 5 vertices, with an arbitrary 2-list assignment, $\{L(v) : v \in V(G)\}$. Suppose G has a vertex, v, of degree 1, with neighbor w. We apply induction to $G - v$ and then color v with a color from $L(v)$ different from the color used for w.

Now assume G has no vertex of degree 1. By the lemma, there exists two adjacent vertices, u and v, of degree two. Let $N(u) = \{a, v\}$ and $N(v) = \{u, b\}$. Then the graph $G' = G \setminus \{u, v\}$ is also outerplanar and triangle-free with list assignment $L' = L|_{v \in V(G')}$. Let $c' : V(G') \to \mathbf{Z}^+$ be a 1-defective L'-list coloring of G'.

Now we obtain c, a 1-defective L-list coloring for G by assigning $c(u) \in L(u) \setminus \{c'(a)\}$ and $c(v) \in L(v) \setminus \{c'(b)\}$ and $c(x) = c'(x)$, for $x \neq u, v$. Then $c(u) \neq c(a)$ and $c(v) \neq c(b)$, and even if $c(u) = c(v)$ then $def_c(u) = def_c(v) = 1$. □

Corollary 1 *All triangle-free outerplanar graphs are (2,1)-colorable.*

Proof: Use Theorem 2, with the added condition that all the lists $L(v)$ are the same for all vertices v. □

Corollary 2 *If G is an outerplanar graph with a 2-list assignment L and two precolored vertices u and v, such that uv is an edge in the outer circuit of G, then this coloring can be extended to a 2-defective L-list coloring of G such that*

- *if $c(u) = c(v)$ then $def(u) = def(v) = 1$ and*
- *if $c(u) \neq c(v)$ then $def(u) = 0$ and $def(v) \leq 1$.*

Proof: This immediately follows as a corollary to Theorem 1 □

5 Bipartite Planar Graphs

From Alon and Tarsi ([1]), we have that all bipartite planar graphs are (3,0)-choosable. The proof is nontrivial. Unexpectedly, if we restrict ourselves to lists of size 2, there is no d such that this class of graphs is $(2, d)$-choosable. This situation is illustrated in Table 3.

Theorem 3 *There is no d such that all bipartite planar graphs are $(2, d)$-choosable.*

Proof: Let $d \geq 0$. Set $A = \{a_1, ..., a_{2d+1}\}$, $B = \{b_1, ..., b_{2d+1}\}$, $C = \{c_1, ..., c_{2d+1}\}$, and $D = \{d_1, ..., d_{2d+1}\}$. We construct a graph G_d as follows: let

$$V(G_d) = A \cup B \cup C \cup D \cup \{u, v\}$$

colors	defects 0	1	2	3
2	X	X	X	X
3	O	O	O	O

Table 3: Defect choosability of bipartite planar graphs

$$\text{and } E(G_d) = \{\{u, w\}, \{w, v\} : w \in A \cup B \cup C \cup D\}.$$

We construct a 2-list assignment L as follows: $L(u) = \{1, 2\}$, $L(v) = \{3, 4\}$, and for $i = 1, ..., 2d + 1$, $L(a_i) = \{1, 3\}$, $L(b_i) = \{1, 4\}$, $L(c_i) = \{2, 3\}$, and $L(d_i) = \{2, 4\}$.

We argue that G_d is not d-defective L-list choosable. Indeed, suppose it were and that c was such a defective coloring. Assume without loss of generality that $c(u) = 1$ and $c(v) = 3$. Then at least $d + 1$ vertices in A are all colored 1 or all colored 3, both of which cases produce a vertex with defect $d + 1$, a contradiction which proves the theorem. □

6 Acknowledgment

We have recently learned that similar results to those presented in this paper are also given in as yet unpublished work by R. Škrekovski, [6]. He reaches the same conclusions but his proofs are quite different. His work also gave us the inspiration to strengthen our Theorem 1.

References

[1] N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica, Vol. 12, No. 2 (1992), 125-134.

[2] L.J. Cowen, R.H. Cowen, D.R. Woodall, Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency, Journal of Graph Theory, Vol. 10 (1986), 187-195.

[3] P. Erdös, A. Rubin, H. Taylor, Choosability in graphs, Congressus Numerantium, Vol. 26 (1979), 125-157.

[4] T. Hull, Some problems in list coloring bipartite graphs, Ph.D. Thesis, Department of Mathematics, University of Rhode Island, Kingston, RI, 1997.

[5] M. Mirzakhani,, A small non-4-choosable planar graph, Bulletin of the ICA, Vol. 17 (1996), 15-18.

[6] R. Škrekovski, List improper colorings of planar graphs, Manuscript.

[7] C. Thomassen, Every planar graph is 5-choosable, Journal of Combinatorial Theory (Series B), Vol. 62 (1994), 180-181.

[8] V.G. Vizing, Coloring the vertices of a graph in prescribed colors, Diskret. Analiz. No. 29, Metody Diskret. Anal. Teorii Kodov Shem 101 (1976), 3-10 (in Russian).

[9] M. Voigt, List colorings of planar graphs, Discrete Mathematics, Vol. 120 (1993), 215-219.

[10] M. Voigt, A not 3-choosable planar graph without 3-cycles, Discrete Mathematics, Vol. 146 (1995), 325-328.

[11] D.B. West, Introduction to Graph Theory, Prentice-Hall, Inc., 1996.