

The Combinatorics of Flat Folds: a Survey*

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Abstract

We survey results on the foldability of flat origami models. The main topics are the question of when a given crease pattern can fold flat, the combinatorics of mountain and valley creases, and counting how many ways a given crease pattern can be folded. In particular, we explore generalizations of Maekawa's and Kawasaki's Theorems, develop a necessary and sufficient condition for a given assignment of mountains and valleys to fold up in a special case of single vertex folds, and describe recursive formulas to enumerate the number of ways that single vertex in a crease pattern can be folded.

1 Introduction

It is safe to say that in the study of the mathematics of origami, flat origami has received the most attention. To put it simply, a *flat origami model* is one which can be pressed in a book without (in theory) introducing new creases. We say “in theory” because when one actually folds a flat origami model, slight errors in folding will often make the model slightly non-flat. In our analysis, however, we ignore such errors and assume all of our models are perfectly folded. We also assume that our paper has zero thickness and that our creases have no width. It is surprising how rich the results are using a purely combinatorial analysis of flat origami. In this paper we introduce the basics of this approach, survey the known results, and briefly describe where future work might lie.

First, some basic definitions are in order. A *fold* refers to any folded paper object, independent of the number of folds done in sequence. The *crease pattern* of a fold is a planar embedding of a graph which represents the creases that are used in the final folded object. (This can be thought of as a structural blueprint of the fold.) Creases come in two types: *mountain creases*, which are convex, and *valley creases*, which are concave (see Figure

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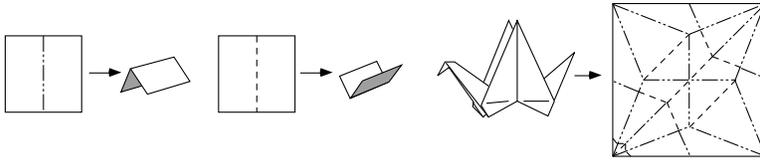


Figure 1: Mountain creases, valley creases, and the crease pattern for the flapping bird with MV assignment shown.

1). Clearly the type of a crease depends on which side of the paper we look at, and so we assume we are always looking at the same side of the paper.

We also define a *mountain-valley (MV) assignment* to be a function mapping the set of all creases to the set $\{M, V\}$. In other words, we label each crease mountain or valley. MV assignments that can actually be folded are called *valid*, while those which do not admit a flat folding (i.e. force the paper to self-intersect in some way) are called *invalid*.

There are two basic questions on which flat-folding research has focused:

1. Given a crease pattern, without an MV assignment, can we tell whether it can flat fold?
2. If an MV assignment is given as well, can we tell whether it is valid?

These are also the focus of this survey. We will not discuss the special cases of flat origami tessellations, origami model design, or other science applications.

2 Classic single vertex results

We start with the simplest case for flat origami folds. We define a *single vertex fold* to be a crease pattern (no MV assignment) with only one vertex in the interior of the paper and all crease lines incident to it. Intersections of creases on the boundary of the paper clearly follow different rules, and nothing of interest has been found to say about them thus far (except in origami design; see [14], [15]). A single vertex fold which is known to fold flat is called a *flat vertex fold*. We present a few basic theorems relating to necessary and sufficient conditions for flat-foldability of single vertex folds. These theorems appear in their cited references without proof. While Kawasaki, Maekawa, and Justin undoubtedly had proofs of their own, the proofs presented below appear in [3].

Theorem 2.1 (Kawasaki [10], Justin [5], [6]) *Let v be a vertex of degree $2n$ in a single vertex fold and let $\alpha_1, \dots, \alpha_{2n}$ be the consecutive angles*

between the creases. Then v is a flat vertex fold if and only if

$$\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{2n} = 0. \quad (1)$$

Proof: Consider a simple closed curve which winds around the vertex. This curve mimics the path of an ant walking around the vertex on the surface of the paper after it is folded. We measure the angles the ant crosses as positive when traveling to the left and negative when walking to the right. Arriving at the point where the ant started means that this alternating sum is zero. The converse is left to the reader; see [3] for more details. \square

Theorem 2.2 (Maekawa, Justin [6], [8]) *Let M be the number of mountain creases and V be the number of valley creases adjacent to a vertex in a single vertex fold. Then $M - V = \pm 2$.*

Proof: (Siwanowicz) If n is the number of creases, then $n = M + V$. Fold the paper flat and consider the cross-section obtained by clipping the area near the vertex from the paper; the cross-section forms a flat polygon. If we view each interior 0° angle as a valley crease and each interior 360° angle as a mountain crease, then $0V + 360M = (n - 2)180 = (M + V - 2)180$, which gives $M - V = -2$. On the other hand, if we view each 0° angle as a *mountain* crease and each 360° angle as a *valley* crease (this corresponds to flipping the paper over), then we get $M - V = 2$. \square

In the literature, Theorem 2.1 and 2.2 are referred to as Kawasaki's Theorem and Maekawa's Theorem, respectively. Justin [7] refers to equation (1) as the *isometries condition*. Kawasaki's Theorem is sometimes stated in the equivalent form that the sum of alternate angles around v equals 180° , but this is only true if the vertex is on a flat sheet of paper. Indeed, notice that the proofs of the Kawasaki's and Maekawa's Theorems do not use the fact that $\sum \alpha_i = 360^\circ$. Thus these two theorems are also valid for single vertex folds where v is at the apex of a cone-shaped piece of paper. We will require this generalization in sections 4 and 5.

Note that while Kawasaki's Theorem assumes that the vertex has even degree, Maekawa's Theorem does not. Indeed, Maekawa's Theorem can be used to prove this fact. Let v be a single vertex fold that folds flat and let n be the degree of v . Then $n = M + V = M - V + 2V = \pm 2 + 2V$, which is even.

3 Generalizing Kawasaki's Theorem

Kawasaki's Theorem gives us a complete description of when a single vertex in a crease pattern will (locally) fold flat. Figure 2 shows two examples of crease patterns which satisfy Kawasaki's Theorem at each vertex, but

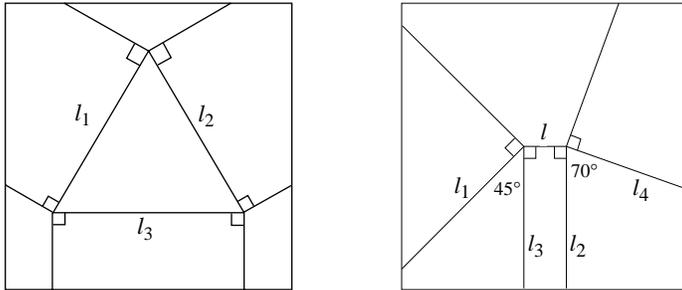


Figure 2: Two impossible-to-fold-flat folds.

which will not fold flat. The example on the left is from [3], and a simple argument shows that no two of the creases l_1, l_2, l_3 can have the same MV parity. Thus no valid MV assignment for the lines l_1, l_2, l_3 is possible. The example on the right has valid MV assignments, but still fails to fold flat. The reader is encouraged to copy this crease pattern and try to fold it flat, which will reveal that some flap of paper will have to intersect one of the creases. However, if the location of the two vertices is changed relative to the border of the paper, or if the crease l is made longer, then the crease pattern *will* fold flat.

This illustrates how difficult the question of flat-foldability is for multiple vertex folds. Indeed, in 1996 Bern and Hayes [1] proved that the general question of whether or not a given crease pattern can fold flat is NP-hard. Thus one would not expect to find easy necessary and sufficient conditions for general flat-foldability.

We will present two efforts to describe general flat-foldability. The first has to do with the realization that when we fold flat along a crease, one part of the paper is being reflected along the crease line to the other side. Let us denote $R(l_i)$ to be the reflection in the plane, \mathbb{R}^2 , along a line l_i .

Theorem 3.1 (Kawasaki [9], [12], Justin [5], [7]) *Given a multiple vertex fold, let γ be any closed, vertex-avoiding curve drawn on the crease pattern which crosses crease lines l_1, \dots, l_n , in order. Then, if the crease pattern can fold flat, we will have*

$$R(l_1)R(l_2)\cdots R(l_n) = I \quad (2)$$

where I denotes the identity transformation.

Although a rigorous proof of Theorem 3.1 does not appear in the literature we sketch here a proof by induction on the number of vertices. In the base case, we are given a single vertex fold, and it is a fun exercise to

show that condition (2) is equivalent to equation (1) in Kawasaki's Theorem (use the fact that the composition of two reflections is a rotation). The induction step then proceeds by breaking the curve γ containing k vertices into two closed curves, one containing $k - 1$ vertices and one containing a single vertex (the k th).

The condition (2) is not a sufficient condition for flat-foldability (the crease patterns in Figure 2 are counterexamples here as well). In fact, as the induction proof illustrates, Theorem 3.1 extends Kawasaki's Theorem to as general a result as possible.

In [7] Justin proposes a necessary and sufficient condition for general flat-foldability, although as Bern and Hayes predicted, it is not very computationally feasible. To summarize, let C be a crease pattern for a flat origami model, but for the moment we are considering the boundary of the paper as part of the graph. If E denotes the set of edges in C embedded in the plane, then we call $\mu(E)$ the *f-net*, which is the image of all creases and boundary of the paper after the model has been folded. We then call $\mu^{-1}(\mu(E))$ the *s-net*. This is equivalent to imagining that we fold carbon-sensitive paper, rub all the crease lines firmly, and then unfold. The result will be the *s-net*.

Justin's idea is as follows: Take all the faces of the *s-net* which get mapped by μ to the same region of the *f-net* and assign a *superposition order* to them in accordance to their layering in the final folded model. One can thus try to fold a given crease pattern by cutting the piece of paper along the creases of the *s-net*, transforming them under μ , applying the superposition order, and then attempting to glue the paper back together. Justin describes a set of three intuitive *crossing conditions* (see [7]) which must not happen along the *s-net* creases during the glueing process if the model is to be flat-foldable – if this can be done, we say that the *non-crossing condition* is satisfied. Essentially Justin conjectures that a crease pattern folds flat if and only if the non-crossing condition holds. Although the spirit of this approach seems to accurately reflect the flat-foldability of multiple vertex folds, no rigorous proof appears in the literature; it seems that this is an open problem.

4 Generalizing Maekawa's Theorem

To extend Maekawa's Theorem to more than one vertex, we define interior vertices in a flat multiple vertex fold to be *up vertices* and *down vertices* if they locally have $M - V = 2$ or -2 , respectively. We define a crease line to be an *interior crease* if its endpoints lie in the interior of the paper (as opposed to on the boundary), and consider any crease line with both endpoints on the boundary of the paper to actually be two crease lines with an interior vertex of degree 2 separating them.

Theorem 4.1 (Hull [3]) *Given a multiple vertex flat fold, let M (resp. V) denote the number of mountain (resp. valley) creases, U (resp. D) denote the number of up (resp. down) vertices, and M_i (resp. V_i) denote the number of interior mountain (resp. valley) creases. Then*

$$M - V = 2U - 2D - M_i + V_i.$$

Another interesting way to generalize Maekawa's Theorem is to explore restrictions which turn it into a sufficiency condition. In the case where all of the angles around a single vertex are equal, an MV assignment with $M - V = \pm 2$ is guaranteed to be valid. This observation can be generalized to sequences of consecutive equal angles around a vertex.

Let us denote a single vertex fold by $v = (\alpha_1, \dots, \alpha_{2n})$ where the α_i are consecutive angles between the crease lines. We let l_1, \dots, l_{2n} denote the creases adjacent to v where α_i is the angle between creases l_i and l_{i+1} (α_{2n} is between l_{2n} and l_1).

If l_i, \dots, l_{i+k} are consecutive crease lines in a single vertex fold which have been given a MV assignment, let $M_{i, \dots, i+k}$ be the number of mountains and $V_{i, \dots, i+k}$ be the number of valleys among these crease lines. We say that a given MV assignment is valid for the crease lines l_i, \dots, l_{i+k} if the MV assignment can be followed to fold up these crease lines without forcing the paper to self-intersect. (Unless these lines include all the creases at the vertex, the result will be a cone.) The necessity portion of the following result appears in [4], while sufficiency is new.

Theorem 4.2 *Let $v = (\alpha_1, \dots, \alpha_{2n})$ be a single vertex fold in either a piece of paper or a cone, and suppose we have $\alpha_i = \alpha_{i+1} = \alpha_{i+2} = \dots = \alpha_{i+k}$ for some $i < k \leq 2n$. Then a given MV assignment is valid for l_i, \dots, l_{i+k+1} if and only if*

$$M_{i, \dots, i+k+1} - V_{i, \dots, i+k+1} = \begin{cases} 0 & \text{when } k \text{ is even} \\ \pm 1 & \text{when } k \text{ is odd.} \end{cases}$$

Proof: Necessity follows by applications of Maekawa's Theorem. If k is even, then the cross-section of the paper around the creases in question might look as shown in the left of Figure 3. If we consider only this sequence of angles and imagine adding a section of paper with angle β to connect the loose ends at the left and right (see Figure 3, left), then we'll have a flat-folded cone which must satisfy Maekawa's Theorem. The angle β adds two extra creases, both of which must be mountains (or valleys). We may assume that the vertex points up, and thus we subtract two from the result of Maekawa's Theorem to get $M_{i, \dots, i+k+1} - V_{i, \dots, i+k+1} = 0$.

If k is odd (Figure 3, right), then this angle sequence, if considered by itself, will have the loose ends from angles α_{i-1} and α_{i+k+1} pointing in the same direction. If we glue these together (extending them if necessary)

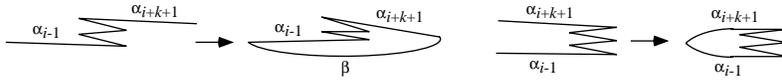


Figure 3: Applying Maekawa when k is even (left) and odd (right).

then Maekawa’s Theorem may be applied. After subtracting (or adding) one to the result of Maekawa’s Theorem because of the extra crease made when gluing the loose flaps, we get $M_{i,\dots,i+k+1} - V_{i,\dots,i+k+1} = \pm 1$.

For sufficiency, we proceed by induction on k . The result is trivial for the base cases $k = 0$ (only one angle, and the two neighboring creases will either be M, V or V, M) and $k = 1$ (two angles, and all three possible ways to assign 2 M’s and 1 V, or vice-versa, can be readily checked to be foldable). For arbitrary k , we will always be able to find two adjacent creases l_{i+j} and l_{i+j+1} to which the MV assignment assigns opposite parity. Let l_{i+j} be M and l_{i+j+1} be V. We make these folds and we can imagine that α_{i+j-1} and α_{i+j} have been fused into the other layers of paper, i.e. removed. The value of $M - V$ will not have changed for the remaining sequence $l_i, \dots, l_{i+j-1}, l_{i+j+2}, \dots, l_{i+k}$ of creases, which are flat-foldable by the induction hypothesis. \square

5 Counting valid MV assignments

We now turn to the question of counting how many different ways we can fold a flat origami model. By this we mean, given a crease pattern that is known to fold flat, how many different valid MV assignments are possible?

We start with the single vertex case. Let $C(\alpha_1, \dots, \alpha_{2n})$ denote the number of valid MV assignments possible for the vertex fold $v = (\alpha_1, \dots, \alpha_{2n})$.

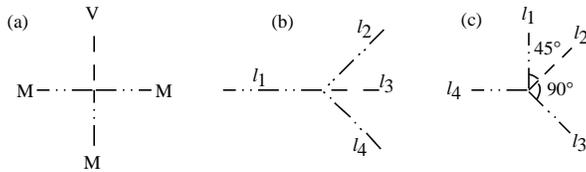


Figure 4: The three scenarios for vertices of degree 4.

An example, consider the case where $n = 2$ (so we have 4 crease lines at v). We compute $C(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ using Maekawa’s Theorem. Its value will depend on the type of symmetry the vertex has, and the three possible situations are depicted in Figure 4. $C(90, 90, 90, 90) = 8$ because any crease could be the “odd crease out” and the vertex could be up or down. In Figure 4 (b) we have only mirror symmetry, and by Theorem

4.2, $M_{2,3,4} - V_{2,3,4} = \pm 1$. Thus l_2, l_3, l_4 must have 2 M's and 1 V or vice versa; this determines l_1 's parity, giving $C(\alpha_1, \dots, \alpha_4) = 6$. In Figure 4 (c) $M_{1,2} - V_{1,2} = 0$, so l_1 and l_2 can be M,V or V,M, and the other two must be both M or both V, giving $C(\alpha_1, \dots, \alpha_4) = 4$.

The example in Figure 4 (a) represents the case with no restrictions. This appears whenever all the angles are equal around v , giving $C(\alpha_1, \dots, \alpha_{2n}) = 2\binom{2n}{n-1}$. The idea in Figure 4 (c), where we pick the smallest angle we see and let its creases be M,V or V,M, can be applied inductively to give the lower bound in the following (see [4] for a full proof):

Theorem 5.1 *Let $v = (\alpha_1, \dots, \alpha_{2n})$ be the vertex in a flat vertex fold, on either a flat piece of paper or a cone. Then*

$$2^n \leq C(\alpha_1, \dots, \alpha_{2n}) \leq 2\binom{2n}{n-1}$$

are sharp bounds.

A formula for $C(\alpha_1, \dots, \alpha_{2n})$ seems out of reach, but using the equal-angles-in-a-row concept, recursive formulas exist to compute this quantity in linear time.

Theorem 5.2 (Hull, [4]) *Let $v = (\alpha_1, \dots, \alpha_{2n})$ be a flat vertex fold in either a piece of paper or a cone, and suppose we have $\alpha_i = \alpha_{i+1} = \alpha_{i+2} = \dots = \alpha_{i+k}$ and $\alpha_{i-1} > \alpha_i$ and $\alpha_{i+k+1} > \alpha_{i+k}$ for some i and k . Then*

$$C(\alpha_1, \dots, \alpha_{2n}) = \binom{k+2}{\frac{k+2}{2}} C(\alpha_1, \dots, \alpha_{i-2}, \alpha_{i-1} - \alpha_i + \alpha_{i+k+1}, \alpha_{i+k+2}, \dots, \alpha_{2n})$$

if k is even, and

$$C(\alpha_1, \dots, \alpha_{2n}) = \binom{k+2}{\frac{k+1}{2}} C(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+k+1}, \dots, \alpha_{2n})$$

if k is odd.

Theorem 5.2 was first stated in [4], but the basic ideas behind it are discussed by Justin in [7].

As an example, consider $C(20, 10, 40, 50, 60, 60, 60, 60)$. Theorem 5.1 tells us that this quantity lies between 16 and 112. But using Theorem 5.2 we see that $C(20, 10, 40, 50, 60, 60, 60, 60) = \binom{2}{1} C(50, 50, 60, 60, 60, 60) = \binom{2}{1} \binom{3}{1} C(60, 60, 60, 60) = \binom{2}{1} \binom{3}{1} 2\binom{4}{1} = 48$.

Not much is known about counting valid MV assignments for flat multiple vertex folds. While there are similarities with work done on counting the number of ways to fold up a grid of postage stamps (see [13], [16], [17]), the questions asked are slightly different. For other work, see [7] and [4].

6 Conclusion

In conclusion, the results for flat-foldability seem to have almost completely exhausted the single vertex case. Open problems exist, however, in terms of global flat-foldability, and very little is known about enumerating valid MV assignments for multiple vertex crease patterns.

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