Classifying Frieze Patterns Without Using Groups

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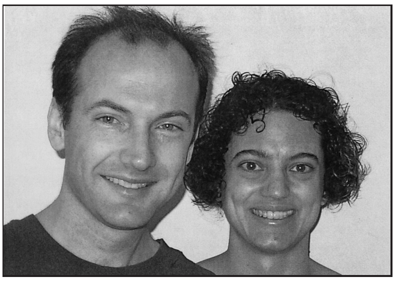

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The authors wrote each other's biographies while teaching at the 2001 Hampshire College Summer Studies in Mathematics, where the accompanying photo was also taken.

Introduction

In classical architecture there's usually a horizontal part (*entablature*) that rests on some columns. The *frieze* is a flat area roughly in the middle of the entablature which can hold a band of ornamentation (Figure 1). The modern equivalent is the wallpaper trim that decorates a foot or so of one's wall near where it meets the ceiling. Usually these designs repeat and thus form a pattern, and in this paper we will classify the different types of frieze patterns or wallpaper trim patterns—we will use the terms interchangeably.

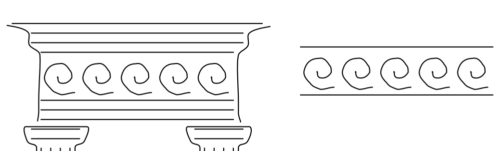

Figure 1. Entablature with frieze vs. wallpaper trim

Many geometry and algebra texts contain proofs or discussions of the fact that there are only seven types of wallpaper trim patterns (e.g., [**1**], [**3**]), where patterns are considered the same type if they have the same rigid symmetries, i.e., *isometries*. All of the proofs known to the authors present this result via group theory, which makes sense as two-dimensional patterns offer a great vehicle for thinking about symmetry groups. However, frieze patterns can also be classified using a combinatorial argument by considering only the distance-preserving transformations that leave the trim invariant. Many institutions (including ours) offer geometry courses for which abstract algebra is not a prerequisite and during which it would not be appropriate to introduce group theory. Transformational geometry is often a topic in these courses and it is useful to be able to discuss frieze patterns in such situations. One could also use frieze patterns in a linear algebra course as an application of affine transformations. In this paper we present a classification scheme for frieze patterns without appealing to group theory.

A combinatorial proof

Our Goal: Prove that there are exactly seven different types of symmetry combinations that can be found in wallpaper trim patterns.

Step 1. Find all types of symmetry a frieze pattern can have.

A wallpaper trim pattern always has, by definition, translational symmetry. Further, any other transformation that leaves the frieze invariant will not involve dilations or skew transformations (which are not isometries), so the only possibilities are rotations, reflections, and glide reflections.

The only nontrivial rotation that brings a frieze back to itself is a 180° rotation. Similarly, the only reflections that leave it invariant are reflections about a vertical or horizontal line. A glide reflection is a translation along a mirror line followed by a reflection over that mirror line. Because we can translate only horizontally, our glide reflections are limited to horizontal translations followed by horizontal reflections. Thus, there are five types of symmetry that a wallpaper trim pattern can have:

t : translation
r : rotation by 180°
h : horizontal reflection
v : vertical reflection
g : glide reflection

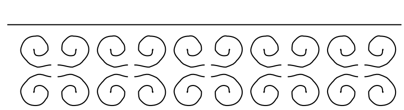

Figure 2. A pattern with all five symmetries

Step 2. Determine which combinations of these symmetries can be found in a wallpaper trim pattern.

First we will list all possibilities, and we will not yet be concerned about whether patterns with these symmetries can exist. Every frieze pattern must have translation symmetry, which leaves the other four symmetry types to choose from. We could choose none, one, two, three, or all four. This gives us

$$1 + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + 1 = 16$$

possible combinations of our symmetries, namely t, r, h, v, g, hv, hr, hg, vr, vg, rg, hvr, hvg, hrg, vrg, and $hvrg$. (See Figure 2 for an example of the last one.)

Not all of these combinations are distinct. For example, composing h and v results in the transformation r, a special case of the well-known fact that the composition of two reflections is a rotation—see [**2**], [**5**]. Therefore, we have

Fact 1. If a trim pattern has h symmetry and v symmetry, then it must also have r symmetry.

We may immediately deduce that the combinations hv and hvg do not represent distinct frieze patterns, and cross them off our list. But let us go through the possibilities more systematically. We will analyze each combination in two phases. Because all of our frieze patterns have translational symmetry, our analysis will begin with a small horizontal segment which produces the entire (infinite) pattern using only translation. We will construct a segment that has all of the symmetries in the combination under consideration, then translate this to form a wallpaper trim pattern and as a second phase consider the symmetries of the entire pattern.

We first consider each of the symmetries r, h, v, and g individually. To study r, let us see (Figure 3) how we could make a wallpaper trim pattern with rotational symmetry using the letter P. (Why P? We chose P because its shape doesn't have any of the symmetries we're trying to examine.)

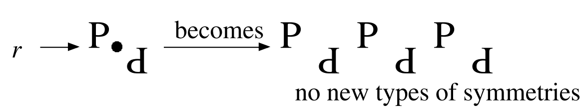

Figure 3. r is a symmetry unto itself

Notice that the resulting pattern does not have h, v, or g symmetry. Notice also that the resulting trim pattern has two types of rotational symmetry; 180° symmetry about a point between two Ps facing each other and about a point between two Ps that are back-to-back. This is a result of composing the transformations t and r. As this second type of rotational symmetry resulted from our generic letter P, it will be present in any frieze pattern with only t and r symmetries. (A good exercise for linear algebra students would be to prove that t and r will always result in another rotational symmetry by using matrices in homogenous coordinates. Alternatively, transformational geometry students could prove this by following the images of two points, say (1, 0) and (0, 1) under r then t, or vice-versa, and observing that the result is a single rotation about a different point.)

Now examine a frieze pattern created from h (Figure 4).

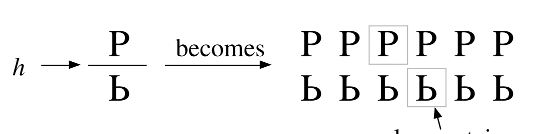

Figure 4. h is never lonely

We see that the existence of the symmetry h in a pattern forces the symmetry g to be present.[1] Notice that while h cannot exist by itself, we have shown that hg does exist, as the picture we drew with P has h and g, but neither r nor v.

The other singletons, illustrated in Figure 5, represent valid combinations of symmetries. (However, notice that when composed with t, the symmetry v will induce another vertical reflection symmetry line. The proof is similar to the analagous situation with r symmetry above.)

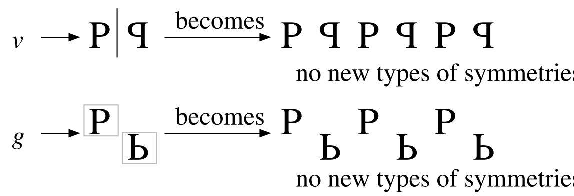

Figure 5. v and g are happy being single

Now consider pairs of h, v, r, g. We've already seen that hv is invalid. In Figure 6 we illustrate hr, which shows that combining h and r produces v and g as well. In the diagram, we produce hr in all possible ways.

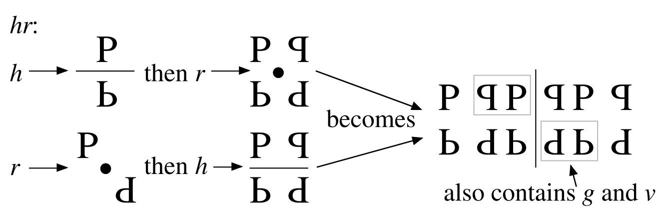

Figure 6. hr says, "We are not alone!"

Fact 2. If a trim pattern has h symmetry and r symmetry, then it must also have v and g symmetry.

We can now rule out hvr and hrg. Further, we observe that $hvrg$ represents a legitimate frieze pattern. We have found six valid wallpaper trim patterns so far: t, v, r, g, hg, and $hvrg$. We still need to check vr, vg, rg, and vrg.

For vr, if we reflect the letter P first then we have two choices of where to place the center of rotation, and if we rotate P first then we have symmetric choices of where to

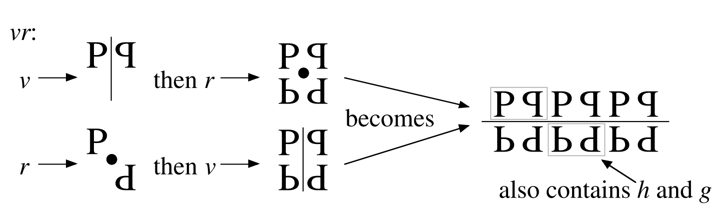

Figure 7. vr produces a familiar pattern

[1]Most treatments of this subject consider a frieze pattern to have glide reflectional symmetry only if this symmetry occurs *within* the smallest horizontal segment which produces the pattern using only translation. We have decided to analyze symmetries of the entire (infinite) pattern, and as a result we see glide reflectional symmetry that others term as *trivial* glide reflectional symmetry. No such distinction is necessary for rotational and reflectional symmetries.

put the line of reflection. If the center of rotation is placed along the vertical axis of symmetry, we will obtain $hvrg$ again (Figure 7).

If the center of rotation is placed off the vertical reflection line, then vrg will result, as illustrated in Figure 8.

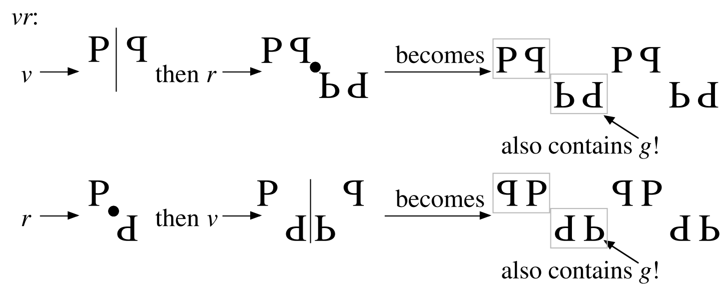

Figure 8. Ooh! This one is new!

Thus we know vrg is a valid wallpaper trim pattern, and we can cross vr off our list. For the symmetry combination vg we see (Figure 9) something similar; the trim pattern must also have r symmetry in this case. (Note that because glide reflections are translational in nature, they must be performed after all other transformations; otherwise, the glide reflectional symmetry will be lost.)

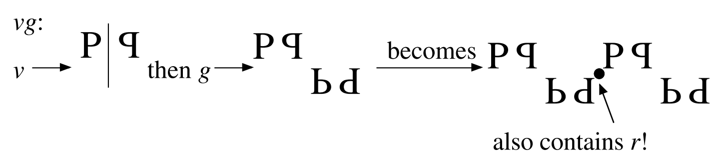

Figure 9. Hey—where did that sneaky r come from?

Finally, we illustrate rg (Figure 10). (We've already shown that hvr, hvg, and hrg are impossible, while vrg is a valid frieze pattern. Our analysis of combinations of three symmetries is complete.)

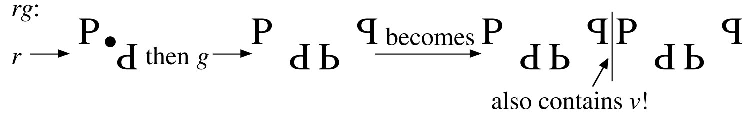

Figure 10. rg falls by the wayside, too...

Thus rg becomes vrg, and we've combined all pairs of h, v, r, and g. This means that we're done, and only seven of the sixteen possible wallpaper trim symmetry combinations are distinct. We summarize our findings in the table below, where "Yes" next to a combination of trim symmetries indicates that this collection is, indeed, distinct from the others.

t		~~hr~~, ~~hvr~~, ~~hrg~~	must also have v
~~h~~	must also have g	~~vr~~	must also have g
v		~~vg~~	must also have r
r		~~rg~~	must also have v
g		vrg	
~~hv~~, ~~hvg~~	must also have r	$hvrg$	
hg			

Table 1. The Final List

Conclusion

The above proof could make an interesting excursion in a geometry or linear algebra course where the concept of group is not assumed. For a great introduction to the group-theoretic approach see Farmer's excellent exposition [**2**].

One might ponder the possibility of using similar methods to prove that there are exactly seventeen different types of wallpaper patterns in the plane. However, the number of combinations we would need to consider increases dramatically to nearly 300, with two linearly independent translation axes, many different kinds of rotation symmetry (it's not hard to prove there are only four; see [**5**]), and many different reflection and glide reflection axes, depending on what other symmetries are present. If such an enumeration based on distance-preserving transformations is possible, it would certainly be as complicated, if not more so, than the group theory based proofs one finds in the literature (see [**6**]).

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