The Flat Vertex Fold Sequences

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1 Introduction

A *flat vertex fold* is a single vertex from a flat origami crease pattern. Almost everything is known about flat vertex folds, but in this paper we investigate an aspect which is not known. In particular, a flat vertex fold is completely determined by the sequence of consecutive angles

$$\vec{v} = (\alpha_1, \alpha_2, ..., \alpha_{2n})$$

about the vertex, which we may think of as a vector. In order to fold flat, however, mountain and valley directions need to be assigned to the creases between these angles, and there are various restrictions on such mountain-valley (MV) assignments. We can then compute the number of *valid* MV assignments for a given vertex $\vec{v}$ and call this number $C(\vec{v})$.

We then ask, "What are the possible values for $C(\vec{v})$ over all flat vertex folds $\vec{v}$ of degree $2n$?" Answering this question leads to some interesting sequences of numbers that shed light into the possibilities of folding a single vertex flat.

In Section 2 we describe, briefly, the basics of flat vertex folds. In Section 3 we outline how to calculate the possible values of $C(\vec{v})$ and begin an initial analysis of the sequences they generate. We conclude the paper with some open questions for further work.

2 The basics of flat vertex folds

The two most fundamental results concerning flat vertex folds are known as Maekawa's Theorem and Kawasaki's Theorem.

Theorem 1 (Maekawa) *Let M and V denote the number of mountain and valley creases, respectively, in a flat vertex fold. Then $M - V = \pm 2$.*

Maekawa's Theorem was independently discovered by Jun Maekawa and Jacques Justin in the 1980s. (See [Kasahara and Takahama 85] and [Justin 86].) Note that it can be used to prove that every flat vertex fold must have an even number of creases.

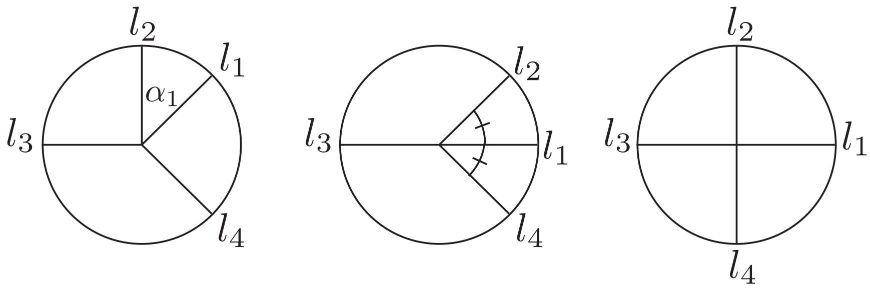

Figure 1: The cases for a degree 4 flat vertex fold.

Theorem 2 (Kawasaki) *Let $\vec{v} = (\alpha_1, ..., \alpha_{2n})$ be a sequence of consecutive angles between creases meeting at a vertex. Then $\vec{v}$ is a flat vertex fold if and only if*

$$\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{2n} = 0.$$

Kawasaki's Theorem was independently discovered by Toshikazu Kawasaki and Jacques Justin in the 1980s. (Again, see [Kasahara and Takahama 85] and [Justin 86].) However, the first reference (and proof) for this result was given by S. A. Robertson [Robertson 78] in 1977, but only for the necessary direction of the Theorem.

It is also interesting to note that these two Theorems are actually special cases of a more general result of Justin's, although it requires us to expand our view to flat-foldable crease patterns with more than one vertex.

Theorem 3 (Justin) *Given a flat origami crease pattern on a simply connected piece of paper, let γ be a vertex-avoiding, simple closed curve on the paper that crosses crease lines $l_1, ..., l_{2n}$, in order. Let α_i be the (oriented) angle between crease lines l_i and l_{i+1} (where we take $l_{2n+1} = l_1$), and let M and V denote the number of mountain and valley creases, respectively, among the creases l_i. Then we have*

$$\alpha_1 - \alpha_2 + \alpha_3 - \cdots - \alpha_{2n} = (M - V)\pi \mod 2\pi.$$

The proof of this is just an easy application of winding numbers (or, if we take some liberties, just the Gauss-Bonnet Theorem). Note, though, that sufficiency is lost; determining if a crease pattern will fold flat globally is notoriously difficult. Also, Justin proved a more general version of this Theorem [Justin 97], showing that it holds for paper with holes as well, although this requires a more complicated proof. We won't be using Justin's Theorem specifically in this paper, but it will be relevant for one of the open problems at the end.

Using Kawasaki's and Maekawa's Theorems as main tools, one can begin an attack on the problem of counting valid MV assignments for flat vertex folds of a specified even degree. (See [Hull 03] for more details.) To offer a motivating example, consider the flat vertex folds in Figure 1. They make up the three canonical cases for a degree four flat vertex fold.

In the first one, creases l_1 and l_2 cannot have the same MV parity, for if they did the two angles surrounding α_1 would both cover it on the same side of the paper, forcing the paper to self-intersect (or force another crease to be made). Maekawa's Theorem then tells us that l_3 and l_4 have to have the same MV parity, and thus we have four possible MV assignments for the creases $l_1, ..., l_4$ ($C(\vec{v}) = 4$ for this vertex).

For the second flat vertex fold in Figure 1, crease lines l_1, l_2, and l_4 cannot all be M or all be V. (If they were all the same, then Maekawa's Theorem would imply that l_3 is different from them, and the two small, equal angles would have to contain the two big angles, which is impossible.) This gives us $C(\vec{v}) = 6$.

In the third example, we have all the angles are equal. Here there are no restrictions on where the Ms and Vs can be–we only have to obey Maekawa's Theorem. This gives us $C(\vec{v}) = 8$.

Notice what is happening in these examples. Whenever we have a sequence of small, equal angles in a row, surrounded by bigger angles, then the creases among those small angles have restrictions placed on them. What is not immediately apparent from the examples in Figure 1 is that we can think of this process as being recursive. If we have a smallest angle (or sequence of consecutive smallest angles), then we can assign MVs to them and fold those creases only, turning our paper into a cone. The creases remaining on this cone can then be looked at by themselves as a separate flat vertex fold (they still must follow Maekawa's and Kawasaki's Theorems).

This insight provides the following bounds:

Theorem 4 *Let $\vec{v} = (\alpha_1, ..., \alpha_{2n})$ be a flat vertex fold, on either a flat piece of paper or a cone. Then*

$$2^n \leq C(\vec{v}) \leq 2\binom{2n}{n-1}$$

are sharp bounds.

The upper bound is achieved by the all-equal-angles case with Maekawa's Theorem applied to it. The lower bound is generated by recursion in the case where there is always one smallest angle to find and assign either MV or VM to its creases, giving us 2^n for a degree $2n$ flat vertex fold.

The question then becomes: What values can $C(\vec{v})$ attain between these bounds? Our recursive process can, surprisingly enough, generate recursive formulas.

Theorem 5 ([Hull 03]) *Let $\vec{v} = (\alpha_1, ..., \alpha_{2n})$ be a flat vertex fold in either a piece of paper or a cone, and suppose we have $\alpha_i = \alpha_{i+1} = \alpha_{i+2} = \cdots = \alpha_{i+k}$ and $\alpha_{i-1} > \alpha_i$ and $\alpha_{i+k+1} > \alpha_{i+k}$ for some i and k. Then*

$$C(\alpha_1, ..., \alpha_{2n}) = \binom{k+2}{\frac{k+2}{2}} C(\alpha_1, ..., \alpha_{i-2}, \alpha_{i-1} - \alpha_i + \alpha_{i+k+1}, \alpha_{i+k+2}, ..., \alpha_{2n})$$

if k is even, and

$$C(\alpha_1, ..., \alpha_{2n}) = \binom{k+2}{\frac{k+1}{2}} C(\alpha_1, ..., \alpha_{i-1}, \alpha_{i+k+1}, ..., \alpha_{2n})$$

if k is odd.

These recursions allow us to compute the various values of $C(\vec{v})$, as will be detailed in the next Section.

3 Flat vertex fold sequences

For a given vertex degree $2n$, the set of possible vectors $\vec{v}$ of dimension $2n$ that could represent a flat vertex fold are restricted by Kawasaki's Theorem. The first author previously described the configuration space P_{2n} of all flat vertex folds of degree $2n$ [Hull 09].

We define

$$SC(n) = \{C(\vec{v}) \mid \vec{v} \in P_{2n}\}.$$

That is, $SC(n)$ is the set of all possible values that $C(\vec{v})$ can attain for flat vertex folds of degree $2n$. We can translate the recursion in Theorem 5 into the sets $SC(n)$, and to do this we will employ the following notation: If x is an integer then let us denote by $xSC(k)$ the set of all elements of $SC(k)$ multiplied by x. (That is, $xSC(k) = \{xa \mid a \in SC(k)\}$.) Then Theorem 5 may be translated as follows:

Theorem 6 *$SC(1) = \{2\}$ and for $n \geq 2$ we have*

$$SC(n) = \left(\bigcup_{k=1}^{n-1} \left(\binom{2n-2k}{n-k} SC(k) \cup \binom{2n-2k+1}{n-k} SC(k) \right) \right) \cup \left\{ 2\binom{2n}{n-1} \right\}.$$

This allows us to easily create the sets $SC(n)$. Below are the first five such sets.

$$\begin{aligned}
SC(1) &= \{2\} \\
SC(2) &= \{4, 6, 8\} \\
SC(3) &= \{8, 12, 16, 18, 20, 24, 30\} \\
SC(4) &= \{16, 24, 32, 36, 40, 48, 54, 60, 70, 72, 80, 90, 112\} \\
SC(5) &= \{32, 48, 64, 72, 80, 96, 108, 120, 140, 144, 160, 162, 180, 200, 210, \\
&\quad 216, 224, 240, 252, 270, 280, 300, 336, 420\}
\end{aligned}$$

We define the sequence $N_n = |SC(n)|$, the sequence of sizes of the sets $SC(n)$. Using *Mathematica* we computed:

$$(N_n) = (1, 3, 7, 13, 24, 39, 62, 97, 147, 215, 312, 440, 617, 851, 1161, ...)$$

This gives us two patterns to investigate: the sets $SC(n)$ and the sizes N_n of the sets. First, we will discuss some bounds on N_n.

Theorem 6 states that to create the set $SC(n)$ we take all previous sets $SC(k)$ for $k < n$, multiply each of them by a binomial coefficient, union them together, and union one more number. If the recursion resulted in all distinct values for $SC(n)$, then the size of the union of all the sets would be equal to the sum of the sizes of the sets being unioned. However, it turns out that the binomial coefficients and the elements in all previous sets $SC(k)$ for $k < n$ multiply to many repeated elements.

For example, in $SC(3)$ the recursion gives us

$$\begin{aligned} SC(3) &= \binom{2}{1}SC(2) \cup \binom{3}{1}SC(2) \cup \binom{4}{2}SC(1) \cup \binom{5}{2}SC(1) \cup \{2\binom{6}{2}\} \\ &= \{8, 12, 16\} \cup \{12, 18, 24\} \cup \\ &\quad \{12\} \cup \{20\} \cup \{30\} \\ &= \{8, 12, 16, 18, 20, 24, 30\}. \end{aligned}$$

We define $SC(k)_i$ to be the ith element in $SC(k)$ ordered least to greatest. In this case, note that $\binom{2}{1}SC(2)_2 = \binom{3}{1}SC(2)_1 = \binom{4}{2}SC(1)_1$; the element 12 has been triple counted. Still, if we disregard the repeats for now, we can use Theorem 6 to easily find an upper bound for N_n. Let a_n be a sequence defined by the recursion

$$a_n = 1 + \sum_{k=1}^{n-1} 2a_k, \text{ where } a_1 = 1. \tag{1}$$

Then a_n is counting exactly what $|SC(n)|$ would be if there were no duplicate numbers ever appearing in the Theorem 6 recursion. Thus a_n is over-counting and we have $N_n \leq a_n$ for all n. After expanding the recursion (1) we have

$$a_n = 2a_{n-1} + a_{n-1} = 3a_{n-1},$$

and therefore $a_n = 3^{n-1}$. Thus we have that $N_n \leq 3^{n-1}$ for all n.

Note that this bound loses accuracy very quickly. Let's compare N_n and a_n for $n = 1, 2, ..., 10$:

$$\begin{aligned} N_n &: \quad 1, 3, 7, 13, 24, 39, 62, 97, 147, 215, ... \\ a_n &: \quad 1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, ... \end{aligned}$$

By $n = 10$, $a_n - N_n = 19648$. Fortunately, we can do better. Let us look for some patterns in the generation of the sets $SC(n)$, using the recursion in Theorem 6. Look again at the recursion for computing $SC(3)$ that we detailed above, and compare it to the recursion for $SC(4)$:

$$\begin{aligned}
SC(4) &= \binom{2}{1}SC(3) \cup \binom{3}{1}SC(3) \cup \\
&\quad \binom{4}{2}SC(2) \cup \binom{5}{2}SC(2) \cup \\
&\quad \binom{6}{3}SC(1) \cup \binom{7}{3}SC(1) \cup \{2\binom{8}{3}\} \\
&= \{16, 24, 32, 36, 40, 48, 60\} \cup \{24, 36, 48, 54, 60, 72, 90\} \cup \\
&\quad \{24, 36, 48\} \cup \{40, 60, 80\} \cup \\
&\quad \{40\} \cup \{70\} \cup \{112\} \\
&= \{16, 24, 32, 36, 40, 48, 54, 60, 70, 72, 80, 90, 112\}.
\end{aligned}$$

You can see a pattern in these examples. In each level of the recursion, the first set is a subset of the second set on the previous line. For example, $\binom{6}{3}SC(1) \subset \binom{5}{2}SC(2)$ and $\binom{4}{2}SC(2) \subset \binom{3}{1}SC(3)$ in the computation of $SC(4)$. If we look at the way the recursion generates these binomial coefficients, we can see that some repeats will always happen.

Looking at the generation of $SC(4)$: The set $\{24, 36, 48\}$ comes from multiplying the elements of $SC(2)$ by $\binom{4}{2} = 6$. Note, though, that $\{24, 36, 48\} \subset \{24, 36, 48, 54, 60, 72, 90\}$. That is because the set $\{24, 36, 48, 54, 60, 72, 90\}$ comes from multiplying the elements of $SC(3)$ by $\binom{3}{1} = 3$. If you consider the fact that some of the elements in $SC(3)$ come from $2 * SC(2)$, then what you have are $3 * (2 * SC(2))$, meaning the elements of $6 * SC(3)$ are purely repeats, mainly because $\binom{4}{2} = 6 = 2\binom{3}{1}$. Similarly, the fact that $\binom{5}{2} = 10$, and $\binom{6}{3} = 20 = 2\binom{5}{2}$ explains why $\binom{6}{3}SC(1) \subset \binom{5}{2}SC(2)$.

This persists throughout the recursion beyond this one example, and we can prove it!

Lemma 7 *Using the notation of Theorem 6, where $1 \leq k \leq n-1$, we have*

$$\binom{2n-2k+2}{n-k+1}SC(k-1) \subset \binom{2n-2k+1}{n-k}SC(k).$$

Proof. Note that

$$\binom{2n-2k+2}{n-k+1} = \frac{(2n-2k+2)(2n-2k+1)!}{(n-k+1)(n-k)!(n-k+1)!} = 2\frac{(2n-2k+1)!}{(n-k)!(n-k+1)!} = 2\binom{2n-2k+1}{n-k}.$$

Also, by the recursion in Theorem 6, we have that $SC(k) = 2SC(k-1) \cup$

(other sets), so $2SC(k-1) \subset SC(k)$ is always true. Therefore,

$$\binom{2n-2k+2}{n-k+1}SC(k-1) = 2\binom{2n-2k+1}{n-k}SC(k-1) \subset \binom{2n-2k+1}{n-k}SC(k).$$

□

This means that the recurrence in Theorem 6 is partly redundant. We can express this as the following:

Theorem 8 *If $SC(n)$ is defined as above, then an equivalent recursion to generate the sets $SC(n)$ is*

$$SC(n) = 2SC(n-1) \cup 3SC(n-1) \cup \left(\bigcup_{k=1}^{n-2} \binom{2n-2k+1}{n-k} SC(k)\right) \cup \left\{2\binom{2n}{n-1}\right\}$$

for $n \geq 2$ and $SC(1) = \{2\}$.

We can use this to obtain a better bound on $N_n = |SC(n)|$. Recall that the sequence given by $a_n = 3a_{n-1}$ gave us a very simple bound on N_n by supposing that all the binomial coefficient terms in Theorem 6 were distinct. Theorem 8 tells us that after the first pair, half of all the other terms will be redundant. So a better bounding sequence $a_n \geq N_n$ would be given by

$$a_n = 3a_{n-1} - a_{n-2} - a_{n-3} - \cdots - a_1.$$

This may be simplified:

$$\begin{aligned} a_n &= 3a_{n-1} - 4a_{n-2} + 3a_{n-2} - a_{n-3} - \cdots a_1 \\ &= 3a_{n-1} - 4a_{n-2} + a_{n-1} \\ &= 4a_{n-1} - 4a_{n-2} \end{aligned}$$

where $a_1 = 1$ and $a_2 = 3$. Solving this homogeneous recurrence gives us $a_n = (n+1)2^{n-2}$. Thus we have proven

Theorem 9 $N_n \leq (n+1)2^{n-2}$ *for all $n \geq 1$.*

This bound can certainly be improved, as there are plenty of other repeated numbers in the $SC(n)$ recursion. However, the exponential upper bound makes one suspect that the sequence N_n might be exponential as well. An exponential lower bound would be needed to more accurately characterize the nature of N_n, but finding a decent lower bound appears to be more difficult. The authors have made only preliminary progress on this and hope to develop it in a future report.

4 Conclusion

Readers may wonder whether an explicit formula for N_n can be found, but this seems highly unlikely. Doing this from the recursions in Theorems 5 and 6 would require knowing the prime factorizations of $\binom{2n}{n}$ and $\binom{2n}{n-1}$. Such prime factorizations are unknown and have been studied extensively (see, for example, [Erdös et al. 75]). However, it is entirely possible that what knowledge does exist about the prime factorization of $\binom{2n}{n}$ could lead to a more comprehensive strategy for avoiding repeated numbers in our recursions, and thus result in better bounds.

Another avenue for future work might be in trying to generalize from the single vertex case towards cases where Justin's Theorem would apply. That is, Justin defines a *crown of faces* C_γ to be all the faces in a flat origami crease pattern that are crossed by a simple, closed, vertex-avoiding curve γ [Justin 97]. Such a crown of faces would form a ring and thus can be viewed as similar to a flat vertex fold, but two problems arise: (1) The creases no longer meet at a point, so the "equal angles in a row" results in the single vertex case that allowed us to count valid MV assignments will no longer apply, and (2) the distances between consecutive creases in our crown of faces will restrict MV assignments further. No results on counting valid MV assignments for crowns of faces are known to the authors, so this could be an interesting and fruitful open area to pursue.

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