

Folding Regular Heptagons

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1 Introduction

The number seven has been historically problematic in the study of constructing regular polygons. Using a straightedge and compass to draw a regular heptagon with mathematical precision is, alongside angle trisections and cube doublings, one of classic Greek geometric construction problems. Mathematicians have known for a few hundred years now that such constructions are impossible to do with these tools. On the other hand, for less than a hundred years mathematicians have known that origami (paper folding) was capable of conquering these construction conundrums.

As tribute to the seventh Gathering 4 Gardner, we will present the theory behind origami regular heptagon construction as well as how to implement it in two different ways.

2 The theory behind the folds

As far as anyone knows, the first person to develop a folding method for the regular heptagon and commit it to paper was Benedetto Scimemi in [Sci89]. His approach uses a standard analysis of the heptagon in the complex plane, as can be seen in Andrew Gleason's paper [Gle88].

The idea is to think of each of the regular heptagon's corners as points in the complex plane corresponding to the 7th roots of unity, that is, solutions to the complex equation $z^7 - 1 = 0$. (See Figure 1.) Of course, $z = 1$ will be one such root, but so will $e^{2\pi i/7}$, since raising this number to the seventh power is the same as adding the angle $2\pi/7$ to itself seven times, giving us

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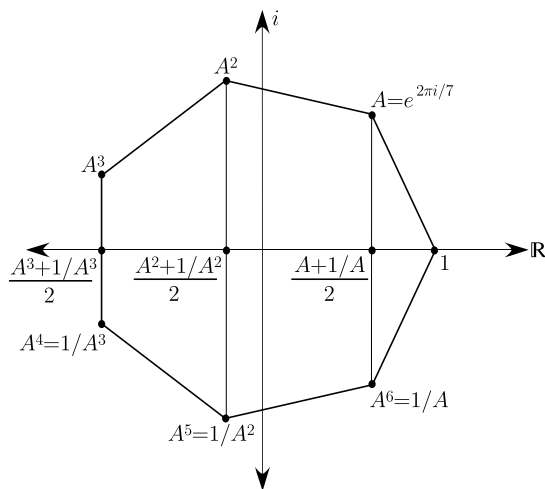


Figure 1: The 7th roots of unity in the complex plane form a heptagon.

$e^{2\pi i} = 1$. If we let $A = e^{2\pi i/7}$, we see that any power of A will also be a solution of $z^7 - 1 = 0$. Thus we also have solutions

$$A^2 = e^{2 \cdot \frac{2\pi}{7}}, A^3 = e^{3 \cdot \frac{2\pi}{7}}, A^4 = e^{4 \cdot \frac{2\pi}{7}}, A^5 = e^{5 \cdot \frac{2\pi}{7}}, A^6 = e^{6 \cdot \frac{2\pi}{7}},$$

and $A^7 = 1$ gets us back to where we started and no more roots will be found. These seven points constitute seven equally-spaced points on the unit circle in the complex plane.

Our job, then, is to locate these points in a sheet of paper via folding. From a folding point of view, if we could just locate the point A , then assuming we already have the point 1 constructed on the real axis, we would have the angle $2\pi/7$ constructed, which could then be copied around the origin to produce the rest of the heptagon. From a mathematical point of view, we're really trying to solve the equation $z^7 - 1 = 0$, and factoring out the root given by $(z - 1)$ gives us

$$\frac{z^7 - 1}{z - 1} = z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0. \quad (1)$$

We can convert this 6th degree equation to a more simple one using the symmetry of our heptagon. Since $A = e^{2\pi i/7} = \cos(2\pi/7) + i \sin(2\pi/7)$,

$$\frac{1}{A} = \frac{\bar{A}}{A\bar{A}} = \frac{\bar{A}}{\cos^2(2\pi/7) + \sin^2(2\pi/7)} = \bar{A} = A^6.$$

Therefore, $A + 1/A = A + \overline{A} = 2 \cos(2\pi/7)$. This can be seen geometrically as well; $(A + 1/A)/2$ is the midpoint of the line segment connecting A and $1/A$, which is on the real axis at $\cos(2\pi/7)$. (This is shown in Figure 1.) Since $\cos(2\pi/7)$ will give us the angle we want, this is really all we need, so we'll turn our focus to the quantity $A + 1/A$.

We can also show that $A^5 = 1/A^2$ and $A^4 = 1/A^3$ (again, see Figure 1). This means that equation (1), if we let $z = A$, becomes

$$\frac{1}{A} + \frac{1}{A^2} + \frac{1}{A^3} + A^3 + A^2 + A + 1 = 0. \quad (2)$$

By expanding out $(A + 1/A)^2$ and $(A + 1/A)^3$ and simplifying, we get that

$$A^2 + \frac{1}{A^2} = (A + 1/A)^2 - 2 \text{ and } A^3 + \frac{1}{A^3} = (A + 1/A)^3 - 3(A + 1/A).$$

Substituting these into equation (2), we get

$$(A + 1/A)^3 + (A + 1/A)^2 - 2(A + 1/A) - 1 = 0.$$

Therefore, $A + 1/A = 2 \cos(2\pi/7)$ is a solution to equation

$$z^3 + z^2 - 2z - 1 = 0. \quad (3)$$

Similar machinations show that the other two roots of equation (3) are $2 \cos(4\pi/7)$ and $2 \cos(6\pi/7)$. In other words, equation (3) is the one we want to solve via origami.

3 Folding it

That origami can solve general third degree equations was first discovered by the Italian mathematician Margherita Beloch in the 1930s [Bel36]. She did this by utilizing an origami move that had not been previously considered: Fold two points to two different lines simultaneously. Beloch demonstrated this by outlining a folding method for constructing $\sqrt[3]{2}$, which is needed to solve the classic Greek problem of doubling the cube. This “2 points to 2 lines” origami move can also be utilized to trisect arbitrary angles (see [Jus84] and [Hul96]). It will also allow us to construct a regular heptagon.

To see what fold will solve equation (3), and thus construct $2 \cos(2\pi/7)$, imagine our piece of paper is the infinite real plane. (We'll see how to do this

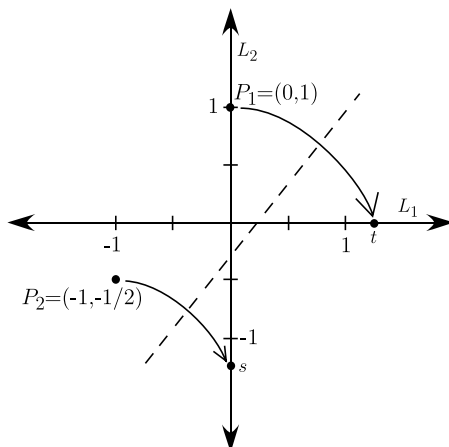


Figure 2: Folding two points to two lines to construct $t = 2 \cos(2\pi/7)$.

in an actual square of paper in a moment.) Label the points $P_1 = (0, 1)$ and $P_2 = (-1, -1/2)$, and let L_1 be the x -axis and L_2 be the y -axis. We then perform Beloch's origami move: Fold the plane so that P_1 lands on L_1 while at the same time making P_2 land on L_2 . Let t be the place on the x -axis where P_1 lands; call this point $P_1' = (0, t)$. (See Figure 2.)

Let's figure out what value t must be. We can do this by determining the equation of our crease line (the dashed line in Figure 2). The segment P_1P_1' has slope $(1-0)/(0-t) = -1/t$, and our crease line will be the perpendicular bisector to this segment. Thus our crease has slope t and the crease must pass through the midpoint of P_1P_1' , which is $(t/2, 1/2)$. Using the point-slope formula for a line, we have that an equation for the crease line is

$$y - 1/2 = t(x - t/2) \Rightarrow y = tx - t^2/2 + 1/2.$$

On the other hand, if we let $P_2' = (0, s)$ be the point where P_2 lands on the y -axis, then the segment P_2P_2' has slope $(2s+1)/2$ and midpoint $(-1/2, (2s-1)/4)$. Thus, another formula for our crease line is

$$y = \frac{-2}{2s+1}x - \frac{1}{2s+1} + \frac{2s-1}{4}.$$

Since these two lines are the same, they must have the same slope. This gives us $s = -(t+2)/(2t)$. But the constant terms of these two line equations must also be equal, that is,

$$-t^2/2 + 1/2 = -1/(2s+1) + (2s-1)/4.$$

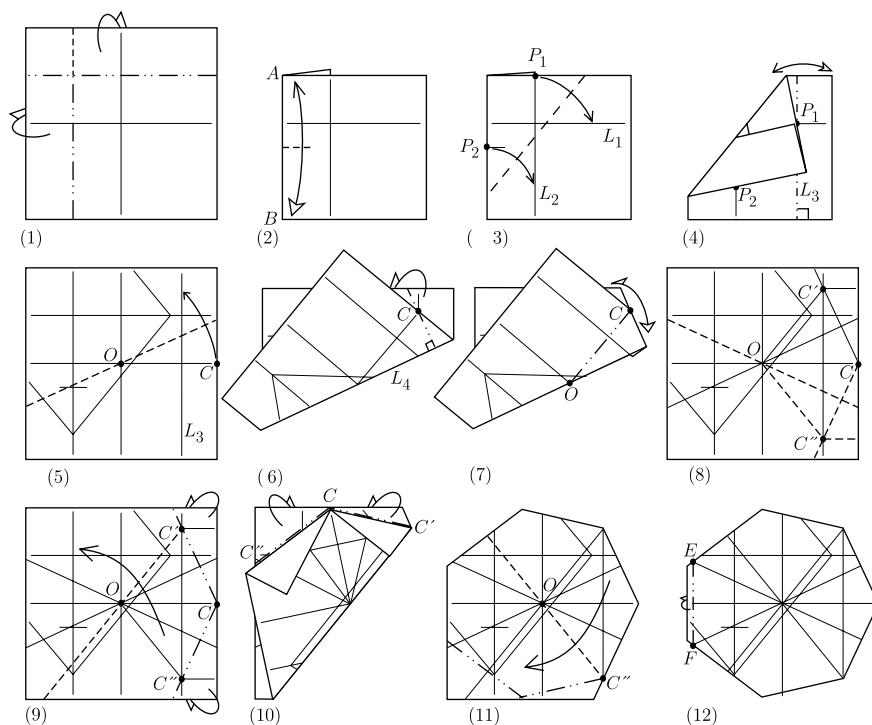


Figure 3: Folding a regular heptagon from a square piece of paper.

If we substitute $s = -(t + 2)/(2t)$ into this and simplify, we'll get a single equation in t :

$$-\frac{t^2}{2} + \frac{1}{2} = \frac{t}{2} - \frac{t+1}{2t} \Rightarrow t^3 + t^2 - t = t + 1 \Rightarrow t^3 + t^2 - 2t - 1 = 0.$$

Lo and behold, we see that t is a solution to equation (3)! Since the other two roots of equation (3) are negative, we have that t must equal $2 \cos(2\pi/7)$.

We really wanted to construct the angle $2\pi/7$, but now this will be easy. Since we have constructed $2 \cos(2\pi/7)$ on the x -axis, all we would need to do is fold a line at the point t perpendicular to the x -axis and use this to make a right triangle whose base is $2 \cos(2\pi/7)$, whose side is the line perpendicular at t , and whose hypotenuse (beginning at the origin) is of length 2. The angle this triangle makes at the origin would then be $2\pi/7$.

All that's left is to implement the origami move in Figure 2 on an actual sheet of paper and do the proper manipulations described in the previous paragraph to create our $2\pi/7$ angle. There are many ways to achieve the

coordinate system in Figure 2 on a sheet of paper. We present one developed by Robert Geretschläger in [Ger97]. Take a square piece of paper and assume that the origin is in the center and that the side of the square is 4 units long. The x - and y - axes can then be constructed by folding the paper in half in both directions, and the points $P_1 = (0, 1)$ and $P_2 = (-1, -1/2)$ can be found as shown in steps (1)-(2) in Figure 3. The “Beloch move” of folding two points to two lines can then be performed as seen in step (3). Since at this stage, the points P_1 and P_2 are on the edges of the folded paper, this move is relatively easy to perform, but it does require bending the paper over and carefully lining up the points before pressing the crease flat.

The other steps in the folding sequence in Figure 3 (which is just a simplification of Geretschläger’s in [Ger97]) are explained as follows:

Step (4): After folding P_1 onto the x -axis (line L_1), it’s location will be at $t = 2 \cos(2\pi/7)$. Folding the vertical line L_3 that passes through this point is making the side of the right triangle described previously.

Step (5): Then unfold everything. We want to make the hypotenuse of length 2 for the right triangle. Since the segment marked OC is of length 2, we can simply fold the paper so that C is placed on line L_3 and the crease passes through O . This is equivalent to using a compass to draw an arc of a circle centered at O and with radius OC . Where C lands on L_3 (call it C') will be the top point of our right triangle. In other words, $\angle C'OC = 2\pi/7$.

Steps (6)-(7): These simply mark where the point C' is and crease the line OC' .

Step (8): Then unfold everything and repeat steps (5)-(7) on the bottom half of the paper to produce the point C'' . So far we have constructed the points equivalent to A and A^6 in Figure 1.

Steps (9)-(12): The rest is simply copying the angles $\angle C'OC$ and $\angle C''OC$ around the origin, while at the same time creasing the other sides of the heptagon.

4 Folding a modular heptagon

Folding a regular heptagon from a single sheet of paper is quite challenging. The folds needed are not trivial, and slight inaccuracies can result in a rather irregular-looking heptagon. With practice, however, it can be done very precisely.

However, we could also use what we learned to make a modular origami

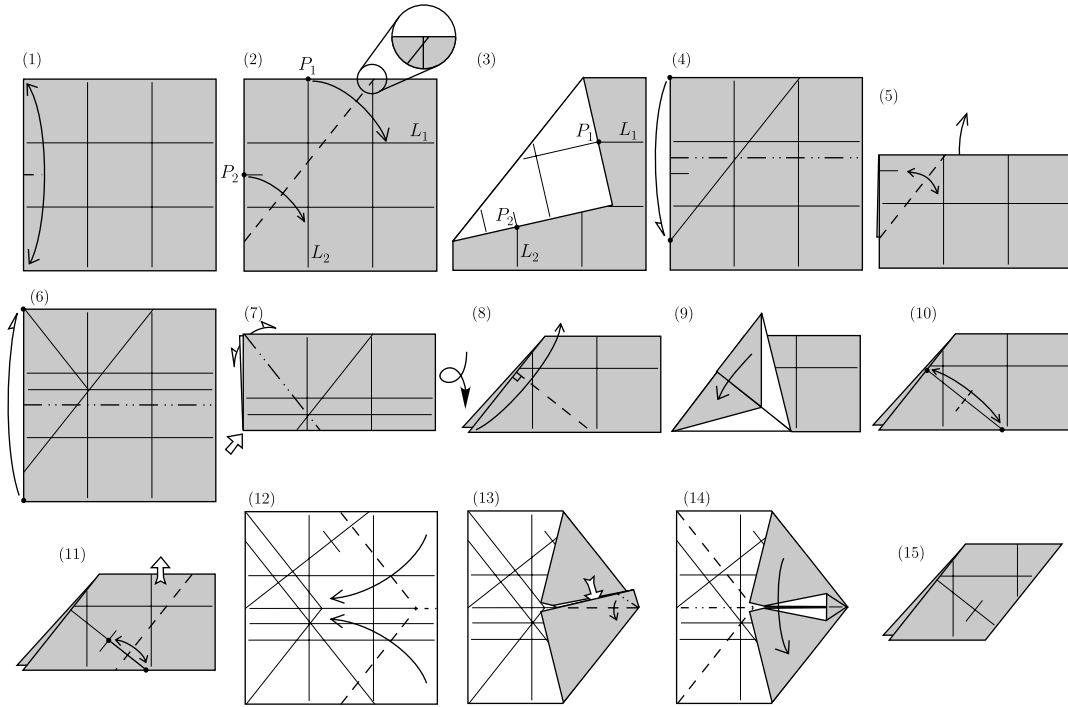


Figure 4: Folding the unit for a modular heptagon ring.

heptagon, and one could argue that this would result in better accuracy. *Modular origami* is where we fold many pieces of paper to make “units” which are then linked together to form a larger, usually geometric object.

There are many ways to modify the folding method for a regular polygon to a modular unit for the same kind of polygon. All that’s needed is the construction of the proper angle. We chose to create a heptagon version of Robert Neale’s classic Magic Ring (aka Pinwheel-Ring) from [Nea94]. Neale, a well-known magic author whose work has appeared in a number of Martin Gardner columns, is also a famous origamist. His Magic Ring is especially popular because it can transform from an 8-sided pinwheel shape to an octagon ring. His sliding and locking mechanisms are especially simple and a natural choice for adaptation to other polygons.

Figure 4 shows the folding sequence for the heptagon ring unit. This is, of course, much more complicated than Neale’s original. The construction of the $2\pi/7$ angle is done using a different coordinate system than the single-sheet version of Figure 3. For the modular version we choose to let the paper be

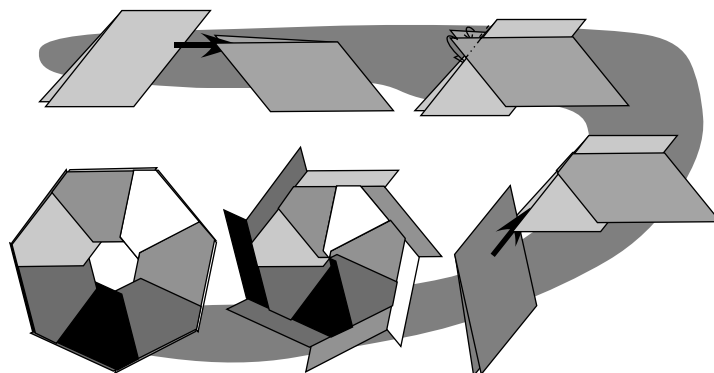


Figure 5: Locking the heptagon ring units together.

three units to a side where the upper right-hand corner has coordinates $(2, 1)$ and the lower left-hand corner is $(-1, -2)$. Thus, the first thing we need to do is to crease the square paper into *thirds* horizontally and vertically. There are a number of ways to do this with mathematical precision, but origamists usually find it easier to just roll the paper into an S-shape and flatten carefully into thirds.

One can then see that step (2) in Figure 4 is the same fold as before: folding $P_1 = (0, 1)$ onto the L_1 (the x -axis) while at the same time bringing $P_2 = (-1, -1/2)$ onto L_2 (the y -axis). This creates the desired angle of $2\pi/7$. The rest of the folding sequence involves moving this angle into the proper place so as to create the structure needed for Neale’s unit.

Seven units need to be folded to form the heptagon ring, and at 15 steps which require a good amount of precision folding, this can be a very time-consuming task. But this can be made much easier as follows: Take care to create one very precise unit using the steps in Figure 4. Then use this as a *template* to make your other units. After all, the finished unit is really just a square piece of paper folded in half and then turned into a parallelogram with two parallel folds at an angle of $2\pi/7$ from the bottom. These two parallel folds can be made very accurately by using the first unit as a guide. One can hold the first unit on top of the in-progress unit to make the needed parallel folds, or partially unfold the first unit and find more clever ways to transfer the angles onto the new unit. Doing this also has the advantage of making units that do not have all the auxiliary creases marring the finished surface.

Instructions for putting the units together to form the ring-pinwheel can be seen in Figure 5. Each unit simply “hugs” a neighboring unit, and you

want to make them hug each other tightly. Putting the last ones in correctly is something of a puzzle, as they do want to slide along their neighbors and the layers of paper become tricky.

5 Conclusion

Scimemi's [Sci89] and Geretschläger's [Ger97] heptagons have been the standard ones known in the origami community. Roger Alperin also provides one in [Alp02], although his method is more complicated. He uses the fact that origami can trisect angles to implement Gleason's straightedge, compass, and angle trisector heptagon construction in [Gle88] on a sheet of paper.

The mathematical theory behind the fact that paper folding can solve general cubic equations is very interesting. Basic expositions of this can be found in [Mar98, Chapter 10] and in Activities 4-6 of [Hul06]. A fun nuts-and-bolts description of how to do it can be found on the web at [Hat03]. A more detailed look at all this from a field extension point of view can be found in another paper of Alperin's [Alp00] as well as in the highly recommended text *Galois Theory* by David Cox [Cox04, Section 10.3].

References

- [Alp00] R. C. Alperin, A mathematical theory of origami constructions and numbers, *New York Journal of Mathematics*, Vol. 6, 2000, 119–133.
- [Alp02] R. C. Alperin, Mathematical origami: another view of Alhazen's optical problem, in *Origami³: Third International Meeting of Origami Science, Mathematics, and Education*, T. Hull, editor, A K Peters, Natick, MA, 2002, 83–93.
- [Bel36] M. P. Beloch, Sul metodo del ripiegamento della carta per la risoluzione dei problemi geometrici, *Periodico di Matematiche*, Series 4, Vol. 16, 1936, 104–108.
- [Cox04] D. Cox, *Galois Theory*, John Wiley & Sons, Hoboken, 2004.
- [Ger97] R. Geretschläger, Folding the regular heptagon, *Crux Mathematicorum*, Vol. 23, No. 2, March 1997, 81–88.

- [Gle88] A.M. Gleason, Angle trisection, the heptagon, and the triskaidecagon, *The American Mathematical Monthly*, Vol. 95, No. 3, 1988, 185–194.
- [Hat03] K. Hatori, Origami construction, 2003, available at <http://origami.ousaan.com/library/conste.html>.
- [Hul96] T. Hull, A note on “impossible” paperfolding, *The American Mathematical Monthly*, Vol. 103, No. 3, 1996, 242–243.
- [Hul06] T. Hull, *Project Origami: Activities for Mathematics Classes*, A K Peters, Natick, MA, 2006.
- [Jus84] J. Justin, described in *British Origami*, No. 107, 1984, 14–15.
- [Mar98] G.E. Martin, *Geometric Constructions*, Springer, New York, 1998.
- [Nea94] R. Neale and T. Hull, *Origami, Plain and Simple*, St. Martin’s Press, New York, 1994.
- [Sci89] B. Scimemi, Draw of a regular heptagon by the folding, in *Proceedings of the First International Meeting of Origami Science and Technology*, H. Huzita, editor, Ferrara, Italy, 1989, 71-77.