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Modelling the folding of paper into three dimensions using affine transformations

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Abstract

We model the folding of ordinary paper via piecewise isometries $\mathbb{R}^2 \to \mathbb{R}^3$. The collection of crease lines and vertices in the unfolded paper is called the crease pattern. Our results generalize the previously known necessity conditions from the more restrictive case of folding paper flat (into $\mathbb{R}^2$); if the crease pattern is foldable, then the product (in a non-intuitive order) of the associated rotational matrices is the identity matrix. This condition holds locally in a multiple vertex crease pattern and can be adapted to a global condition. Sufficiency conditions are significantly harder, and are not known except in the two-dimensional single-vertex case.

Keywords: Affine; Transformations; Homogeneous; Paper; Folding; Non-flat folding

This project was inspired by the intuition that origami (the art of paperfolding) must have some significant mathematical properties. Much of the origami commonly done in the United States folds flat; some origami is then made three-dimensional by manipulating the flat-folded paper to look more lifelike. The most common examples of this phenomenon are the peace crane or flapping bird, and the paper balloon (known in origami circles as the water bomb). Flat-foldability has been analyzed in [2,5]. However, there is nothing either artistically or mathematically which restricts one to folding paper flat. There are origami designs (though comparatively

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few) which require non-flat folds—creases where the dihedral angle is neither 0 nor 2π—for example, the traditional Masu box.

Henceforth in this paper, we will assume that any fold is not necessarily flat. (A *fold* refers to a complete folded object as well as the result of making a single crease.) We will model this type of folding and show necessary conditions for a crease pattern to be foldable. Sufficiency conditions are significantly more difficult and we address this issue in Section 5.

1. Modelling paper-folding: constraints

We wish to model the folding of paper mathematically, by examining a map $f: \mathbb{R}^2 \to \mathbb{R}^3$. In fact, we will use a composition map

$$\mathbb{R}^2 \hookrightarrow \mathbb{R}^3 \to \mathbb{R}^3$$

$$(x, y) \mapsto (x, y, 0) \mapsto (?, ?, ?)$$

Because we think of the paper as sitting in $\mathbb{R}^3$ in this fashion, we will henceforth examine only the $\mathbb{R}^3 \to \mathbb{R}^3$ portion of the above map.

Consider a folded piece of paper; mentally unfold it. As a combinatorial object, it has vertices, edges, and faces. While curved creases are possible (see [4]), in this paper we will only consider creases that are straight lines.

Definition. A *crease pattern* is a simple straight-edged graph embedded in the plane. The edges of this graph correspond to the locations of fold lines in an unfolded sheet of paper. Note that we do not include the boundary of the paper in the crease pattern.

We may begin by thinking of the mechanics of putting a crease into paper. In reality, we hold part of the paper stationary, and fold the rest of the paper flat along the proposed crease line. If we desire the result to be non-flat, then we change the dihedral angle δ of the crease to be greater than zero. Mathematically, we can accomplish the same result by applying a rotation by $\pi - \delta$ radians, which we call the *folding angle*, to the portion of the plane corresponding to the part of the paper we lifted to make our crease. This sets the stage for the following analysis.

Note the following:

(1) Paper does not stretch.
(2) The faces of the folded paper are flat (as opposed to curved).
(3) We do not want the paper to rip or have holes.
(4) Paper does not intersect itself.

Constraint (2) indicates that f must be piecewise affine; constraint (1) restricts f to a piecewise isometry. In fact, as we observed above, f will be composed only of rotation matrices.

Constraint (3) specifies that f must be continuous.

Constraint (4) is at the heart of any sufficiency condition, and is as yet not completely accounted for—see Section 5.

Definition. A crease pattern is *foldable* if it is physically possible to produce a piece of paper which has crease lines in the indicated positions with dihedral angles as indicated, such that the paper does not stretch, rip, or intersect itself during folding and such that the faces of the resulting fold are flat.

For simplicity, we first consider crease patterns with only a single multi-valent vertex. (We will abuse notation by using the term *single-vertex fold*.) The reader may wish to think of this as a local case, in which we examine an open set around a particular vertex of a more complex crease pattern.

2. Analysis of a single-vertex fold

Fig. 1 is an example of a single-vertex fold, which is a common fold found in "box pleated" origami designs. In general the crease lines $l_1, \ldots, l_n$ are enumerated in counterclockwise order beginning from the x-axis. We label each crease line l_i with a pair of angles (α_i, ρ_i), where α_i denotes the position of the crease in the paper (the plane angle) and ρ_i denotes the folding angle. However, in our illustrations we only mark the folding angles because the plane angles are implicit from the drawing of the crease pattern.

We wish to define our map f using only the given information $\{l_i : (\alpha_i, \rho_i)\}$. To ensure continuity of the map, we impose a folding order via a closed path γ travelling counterclockwise around the vertex.

As we traverse the closed path on the crease pattern, we also traverse its image on the completed fold. Travelling along γ on the crease pattern (resp. on the completed fold), two actions happen:

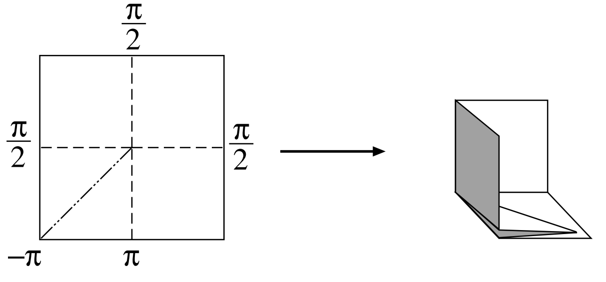

Fig. 1. A labelled crease pattern and the corresponding completed fold. Note that only the folding angles for the creases are labelled.

1. We move across a face of the crease pattern (resp. across a face of the completed fold). This will be accounted for by a change in the *xy* coordinates on the plane of the unfolded paper.
2. We move from from one face to another by crossing a crease line l_i (resp. by rotating around a crease line l_i). In the completed fold, we rotate by the supplement ρ_i of the dihedral angle δ_i between the two faces. Because the fold is a piecewise linear object, we may express this action via a linear map (matrix), and denote it by L_i.

Consider, then, a single-vertex crease pattern on which each crease line l_i is assigned an ordered pair (α_i, ρ_i).

Let A_i be the matrix corresponding to a rotation in the *xy*-plane by the plane (or location) angle α_i. Let C_i be the matrix which rotates by the folding angle ρ_i in the *yz*-plane, i.e. around the *x*-axis counterclockwise. Now, use $\chi_i = A_i C_i A_i^{-1}$ to denote the rotation counterclockwise of angle ρ_i around the axis corresponding to the crease line l_i in the plane.

Note. A_i and C_i (and thus χ_i as well) are orthogonal matrices, which reflect the non-rubbery nature of the paper.

Now we will define the L_i in terms of the χ_j. Certainly $L_1 = \chi_1$, and $L_2 = \chi_1\chi_2\chi_1^{-1}$ because we undo operation L_1 to put crease line l_2 back in the *xy*-plane, then perform fold χ_2, and redo our first crease to put the whole ensemble back. (Recall that the order of matrix multiplication is determined by viewing the process as linear functions applied to a vector in $\mathbb{R}^3$.) Using this type of reasoning, we see that in general L_i=(matrix to redo the previous L's) χ_i (matrix to undo the previous L's in reverse order), so that as a matrix L_i is equal to $(L_{i-1}\cdots L_1)\chi_i(L_1^{-1}\cdots L_{i-1}^{-1})$. This recursive definition gives us that $L_i = \chi_1\cdots\chi_{i-1}\chi_i\chi_{i-1}^{-1}\cdots\chi_1^{-1}$.

We have implicitly defined a piecewise linear map $f : \mathbb{R}^2 \subset \mathbb{R}^3 \to \mathbb{R}^3$ corresponding to our fold as follows. Enumerate the faces of the crease pattern in $\mathbb{R}^2$ as $F_1, \ldots, F_n$ so that face F_j lies between crease lines l_j and l_{j+1}. Explicitly,

$$
\begin{aligned}
f((x, y)) &= f((x, y, 0)) \\
&= \begin{cases}
L_1(x, y, 0) & \text{for } (x, y) \in F_1, \\
L_2L_1(x, y, 0) & \text{for } (x, y) \in F_2, \\
\quad\vdots & \quad\vdots \\
L_{n-1}\cdots L_1(x, y, 0) & \text{for } (x, y) \in F_{n-1}, \\
L_nL_{n-1}\cdots L_1(x, y, 0) = I(x, y, 0) & \text{for } (x, y) \in F_n.
\end{cases}
\end{aligned}
$$

Note that the “last” face of the paper lies in the *xy*-plane, and that this includes both l_1 and l_n. Also, it is easy to redefine the map for a clockwise path; we leave this to the reader.

3. A necessary condition

Given a single-vertex crease pattern with n creases, we may write the accumulation of the operations along a closed counterclockwise path γ as $L_n L_{n-1} \cdots L_2 L_1$. In fact, we see that we are required to have $L_n L_{n-1} \cdots L_2 L_1 = I$ to ensure continuity of the map along the l_n-axis. (This was constraint (3) above, which prevents us from ripping the paper.)

There is another reason why $L_n \cdots L_1 = I$, which will be crucial in Section 4 on multi-vertex crease patterns. Note that there is a subtle difference between physically folded paper and mathematically folded paper (as in our model). A physical fold corresponds to a developable surface (i.e., is isometric to the plane) but mathematically folded paper does not. In our corner fold example (Fig. 1), the points $(-2, -2, 0)$ and $(2, -2, 0)$ are identified in the fold map f even though they are distance 4 apart on the crease pattern.

Despite this difference, we would like to be able to use the fact that the parallel transport of a vector around a closed curve on a developable surface does not change its orientation. Certainly the parallel transport of a vector around a closed curve on the *image* of our crease pattern *can* change the orientation of the vector. (The reader can check this on the corner fold in Fig. 1.) However, instead of examining closed curves on a mathematical fold, we can examine the image under f of a curve γ on the unfolded paper. Because γ lies in the plane and f is composed only of rotations, $f(\gamma)$ is locally isometric to γ, and so the parallel transport of a vector around $f(\gamma)$ will not change its orientation. (This certainly holds for generic γ. In order to assure that $f(\gamma)$ is locally isometric to γ, we may sometimes need to deform γ so that $f(\gamma)$ will not cross itself.) Therefore, because the L_i are rotations, $L_n \cdots L_1$ encodes the effect on the orientation of a vector of parallel transport and so is equivalent to I.

Theorem 3.1. *Given a foldable single-vertex crease pattern with associated folding map as defined above,* $\chi_1 \cdots \chi_{n-1}\chi_n = I$.

Proof. Using our notation from above and substituting our earlier definitions, we see that

$$\begin{aligned} L_n \cdots L_1 &= (\chi_1 \cdots \chi_{n-1}\chi_n \chi_{n-1}^{-1} \cdots \chi_1^{-1})(\chi_1 \cdots \chi_{n-2}\chi_{n-1}\chi_{n-2}^{-1} \cdot \chi_1^{-1}) \\ &\quad \times \cdots (\chi_1 \chi_2 \chi_1^{-1})\chi_1 \\ &= \chi_1 \cdots \chi_{n-1}\chi_n. \end{aligned}$$

This concludes the proof. □

That $\chi_1 \cdots \chi_{n-1}\chi_n = I$ was stated without proof by Kawasaki [8]. After we worked through the details, it became apparent to us that Kawasaki must have observed much of what we present here. We should note that Kawasaki himself states this criterion in the form of a definition: a crease pattern is foldable if and only if

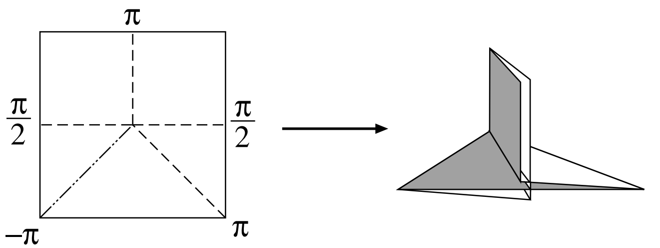

Fig. 2. A fold that satisfies Theorem 3.1 but also self-intersects.

the condition $\chi_1 \cdots \chi_{n-1}\chi_n = I$ holds. (In contrast, recall our earlier definition of *foldable*.)

The converse of this statement is not true—there exist crease patterns such that $\chi_1 \cdots \chi_{n-1}\chi_n = I$ but which are not physically foldable.

Example. Examine the crease pattern $l_1 = (0, \pi/2)$, $l_2 = (\pi/2, \pi)$, $l_3 = (\pi, \pi/2)$, $l_4 = (5\pi/4, -\pi)$, $l_5 = (7\pi/4, \pi)$. The "flap" intersects one of the "walls" (see Fig. 2).

A special case of Theorem 3.1 is the previously known analogous result for single-vertex *flat* folds. Let $\Delta\alpha_i = \alpha_{i+1} - \alpha_i$ denote the angle between crease lines l_{i+1} and l_i. Given a single-vertex flat fold, the folding angles ρ_i will all be $\pm\pi$ and the rotation matrices χ_i will be reflections over the crease lines in the xy plane. Because the composition of two reflections is a rotation by twice the angle between the reflection lines, and we can think of the identity transformation as rotating about the origin by 2π, Theorem 3.1 becomes

$$\Delta\alpha_1 + \Delta\alpha_3 + \cdots + \Delta\alpha_{2k-1} = \Delta\alpha_2 + \Delta\alpha_4 + \cdots + \Delta\alpha_{2k} = \pi,$$

where the crease pattern has $2k = n$ edges. (It is not hard to show that every single-vertex flat fold must have an even number of creases.) This is sometimes stated as $\Delta\alpha_1 - \Delta\alpha_2 + \Delta\alpha_3 - \cdots - \Delta\alpha_{2n} = 0$, and it is a necessary and sufficient condition for a single-vertex fold to fold flat. It was stated without proof by Justin [6] and Kawasaki [7] and first proven in [5].

4. Necessity for multi-vertex crease patterns

The discussion of multi-vertex crease patterns requires some new notation. Here, we only consider bounded paper; in theory, there is no problem with infinite paper as long as the folding map is defined everywhere in the plane.

Consider a crease pattern with vertices, edges, and faces, and choose some vertex to lie at the origin. Label this vertex v_0. Enumerate the other vertices in some fashion as v_i, so that there are vertices $v_0, \ldots, v_b$. Some crease lines may be incident to the boundary of the paper; the points of incidence are "virtual vertices", in that for indexing purposes we will want to label them, but they do not act as vertices for the purposes of our results. Denote these points as $(vv)_i$, and enumerate them $(vv)_{b+1}, \dots, (vv)_c$. Now, denote the crease lines as $l_{(i,j)}$ where i, j are the indices corresponding to the incident vertices (one of which may be virtual). Choose one face adjacent to v_0 to be fixed in the *xy* plane, and denote it F_0. Finally, denote the other faces by $F_{(i,j,\dots)}$, where the coordinates of $(i, j, \dots)$ are the indices corresponding to the vertices surrounding each face.

In order to describe the action of folding along crease line $l_{(i,j)}$, we want to describe an affine rotation about the line in which $l_{(i,j)}$ lies; thus, we will need to move to homogeneous coordinates. As before, each crease line $l_{(i,j)}$ is marked with an ordered pair $(\alpha_{(i,j)}, \rho_{(i,j)})$ corresponding to the angle to the *x*-axis and the folding angle. We will accomplish the crease by moving a vertex of $l_{(i,j)}$ to the origin, creasing as before, and returning $l_{(i,j)}$ to its original position. Let B_k be the 4×4 matrix which translates a point in $\mathbb{R}^3$ by v_k; let $A_{(i,j)}$ be the matrix in homogeneous coordinates which rotates the *xy*-plane by angle $\alpha_{(i,j)}$, and let $C_{(i,j)}$ be the matrix in homogeneous coordinates which rotates by angle $\rho_{(i,j)}$ in the *yz*-plane. Then the folding matrix will be $\chi_{((i,j),k)} = B_k A_{(i,j)} C_{(i,j)} A^{-1}_{(i,j)} B^{-1}_k$. Note that k indicates which vertex we consider $l_{(i,j)}$ to extend from, so that $k = i$ or $k = j$.

Now consider a closed vertex-avoiding path γ on the crease pattern, beginning and ending on F_0. Let $l_{(i_t,j_t)}$ be the relevant crease lines, enumerated in the order that γ crosses them. Following the model from Section 2, let $L_{(i_t,j_t)}$ encode the rotation in the completed fold around $l_{(i_t,j_t)}$ by $\rho_{(i_t,j_t)}$. Note that $L_{(i_t,j_t)}$ is a 4×4 matrix in homogeneous coordinates, and dependent on γ. It is recursively defined, as follows: $L_{(i_1,j_1)} = \chi_{((i_1,j_1),k_1)}$, $L_{(i_2,j_2)} = L_{(i_1,j_1)} \chi_{((i_2,j_2),k_2)} L^{-1}_{(i_1,j_1)}$, and in general

$$L_{(i_t,j_t)} = L_{(i_{t-1},j_{t-1})} \cdots L_{(i_1,j_1)} \chi_{((i_t,j_t),k_t)} L^{-1}_{(i_1,j_1)} \cdots L^{-1}_{(i_{t-1},j_{t-1})}.$$

(By convention, we have $k_t =$ the vertex around which we rotate counterclockwise.) This translates to

$$L_{(i_t,j_t)} = \left(\chi_{((i_1,j_1),k_1)} \cdots \chi_{((i_{t-1},j_{t-1}),k_{t-1})}\right) \chi_{((i_t,j_t),k_t)} \times \left(\chi^{-1}_{((i_{t-1},j_{t-1}),k_{t-1})} \cdots \chi^{-1}_{((i_1,j_1),k_1)}\right).$$

Theorem 4.1. *Consider a foldable multi-vertex crease pattern. Let γ be a closed curve on the crease pattern, beginning and ending on face $F_{(i_p,j_p,\dots)}$, which does not intersect any vertices. Let $l_{(i_1,j_1)}, l_{(i_2,j_2)}, \dots, l_{(i_n,j_n)}$ be the creases, in order, that γ crosses, and let $\chi_{((i_t,j_t),k_t)}$ be the rotation counterclockwise around the crease line $l_{(i_t,j_t)}$. Then $\chi_{((i_1,j_1),k_1)} \chi_{((i_2,j_2),k_2)} \cdots \chi_{((i_n,j_n),k_n)} = I$.*

Proof. Let ψ be a vertex-avoiding path from F_0 to $F_{(i_p,j_p,\dots)}$. Then $\Gamma = \psi^{-1}\gamma\psi$ is a closed, vertex-avoiding curve beginning and ending on F_0. Let $L_{(d_1,e_1)}, \dots, L_{(d_m,e_m)}$ be the folding matrices, as defined above, that correspond to the crease lines that Γ crosses, in order. Let Γ' be the image of Γ under these matrices. Using similar reasoning as in Section 3, we see that the parallel transport of a vector around Γ, and thus around Γ', does not affect its orientation. Therefore, as the accumulation of orientation changes along γ' is $L_{(d_m,e_m)} \cdots L_{(d_1,e_1)}$, that transformation must be equivalent to the identity

$$L_{(d_m,e_m)} L_{(d_{m-1},e_{m-1})} \cdots L_{(d_1,e_1)} = I.$$

It then follows, by the same cancellation as in the proof of Theorem 3.1, that

$$\chi_{((d_1,e_1),f_1)} \chi_{((d_2,e_2),f_2)} \cdots \chi_{((d_m,e_m),f_m)} = I.$$

Let $\chi_{((a_1,b_1),c_1)}, \ldots, \chi_{((a_q,b_q),c_q)}$ be the matrices associated with the crease lines that ψ crosses, in order. A short direct computation shows that the matrices associated with ψ^{-1} will be $\chi^{-1}_{((a_q,b_q),c_q)}, \ldots, \chi^{-1}_{((a_1,b_1),c_1)}$, in order. We can now rewrite our identity as

$$\begin{aligned}&\left(\chi_{((a_1,b_1),c_1)} \cdots \chi_{((a_q,b_q),c_q)}\right)\left(\chi_{((i_1,j_1),k_1)} \cdots \chi_{((i_n,j_n),k_n)}\right)\\ &\quad\times\left(\chi^{-1}_{((a_q,b_q),c_q)} \cdots \chi^{-1}_{((a_1,b_1),c_1)}\right) = I.\end{aligned}$$

This has the form $MNM^{-1}=I$, so that $N = I$ or $\chi_{((i_1,j_1),k_1)} \cdots \chi_{((i_n,j_n),k_n)} = I$. □

Theorem 4.1 can also be proven using a dry and technical induction. See [1] for details.

Definition. Let F_0 be the face that is fixed in the *xy* plane. Given any other face $F_{(i_p,j_p,\ldots)}$ let γ be any vertex-avoiding path from a point in F_0 to a point in $F_{(i_p,j_p,\ldots)}$. Let the crease lines that γ crosses be, in order, $l_{(i_1,j_1)}, \ldots, l_{(i_p,j_p)}$. Then the general folding map *f* is

$$\begin{aligned}&f(x,y) = f(x,y,0,1) = \chi_{((i_1,j_1),k_1)} \chi_{((i_2,j_2),k_2)} \cdots \chi_{((i_p,j_p),k_p)}(x,y,0,1)\\ &\quad\text{for } (x,y) \in F_{(i_p,j_p,\ldots)}.\end{aligned}$$

We must now show that *f* is well-defined. This is somewhat surprising, as different choices of γ will result in different combinations of $\chi_{((i_t,j_t),k_t)}$ matrices. It is also interesting that we are able to define *f* only in terms of the $\chi_{((i_t,j_t),k_t)}$ matrices, as they represent rotations about the crease lines $l_{(i_t,j_t)}$ in the *xy* plane.

Theorem 4.2. *The definition of the folding map on face $F_{(i_p,j_p,\ldots)}$ is independent of the defining path γ chosen from the fixed face F_0 in the xy-plane.*

Proof. Consider any two vertex-avoiding paths γ_1 and γ_2 from face F_0 to face $F_{(i_p,j_p,\ldots)}$. Let γ_1 cross (in order) crease lines $l_{(i_1,j_1)}, l_{(i_2,j_2)}, \ldots, l_{(i_m,j_m)}$, and let γ_2 cross (in order) crease lines $l_{(i_n,j_n)}, l_{(i_{n-1},j_{n-1})}, \ldots, l_{(i_{m+1},j_{m+1})}$. The folding map on $F_{(i_p,j_p,\ldots)}$ via γ_1 is $\chi_{((i_1,j_1),k_1)}\ \chi_{((i_2,j_2),k_2)} \cdots \chi_{((i_m,j_m),k_m)}$, and the folding map on $F_{(i_p,j_p,\ldots)}$ via γ_2 is $\chi_{((i_n,j_n),k_n)}\ \chi_{((i_{n-1},j_{n-1}),k_{n-1})} \cdots \chi_{((i_{m+1},j_{m+1}),k_{m+1})}$. If we consider the path $\gamma_1\gamma_2^{-1}$, we see that it is closed and the proof of Theorem 4.1 shows that

$$\chi_{((i_1,j_1),k_1)}\chi_{((i_2,j_2),k_2)}\cdots\chi_{((i_m,j_m),k_m)}\chi^{-1}_{((i_{m+1},j_{m+1}),k_{m+1})}\cdots\chi^{-1}_{((i_n,j_n),k_n)}=I.$$

It then follows that

$$\begin{aligned}&\chi_{((i_1,j_1),k_1)}\chi_{((i_2,j_2),k_2)}\cdots\chi_{((i_m,j_m),k_m)}\\&\quad=\chi_{((i_n,j_n),k_n)}\chi_{((i_{n-1},j_{n-1}),k_{n-1})}\cdots\chi_{((i_{m+1},j_{m+1}),k_{m+1})},\end{aligned}$$

as desired. □

5. Sufficiency and future work

What criteria, in addition to $\chi_{((i_1,j_1),k_1)}\chi_{((i_2,j_2),k_2)}\cdots\chi_{((i_n,j_n),k_n)}=I$, are sufficient for a crease pattern to be foldable? Ideally, we would like such criteria to be testable on examples as well as clearly sufficient.

The only constraint not accounted for by our necessity conditions is that we do not want the paper to self-intersect. The example we considered earlier in Fig. 2 is a situation we want to avoid; Fig. 3 shows two other common types of self-intersection. Less obviously, there can be problems when many layers of paper lay flat. See Fig. 4 for a contrast between a legitimate fold and a self-intersecting fold where multiple faces map to the same area. This latter problem can be reduced to a question of sufficiency for flat-foldability; in the single-vertex case, the criteria were described in Section 3 (see [5–7]). However, this question is open for the multi-vertex case. In other flat-foldability work (see [3]), the issue is avoided.

Concentrating only on non-flat self-intersections, and only on single-vertex folds, we can easily think of characterizations of sufficiency. For example, we could consider a path γ surrounding our vertex, and examine its image $f(\gamma)$; it will be equivalent to the unknot, but only if no Reidemeister moves are necessary will there be no self-intersections. Or, consider the paper as having color on one side and not on the other (as origami paper often does). When folded, is the exterior of the completed fold completely colored and are the interior regions of the completed fold completely blank? If so, there are no self-intersections. We could certainly examine the faces of

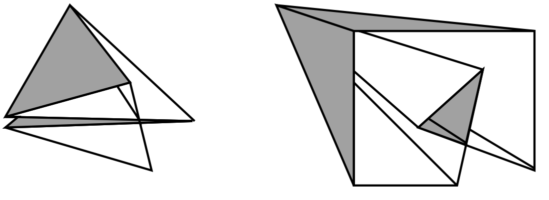

Fig. 3. Some other ways that the paper can self-intersect.

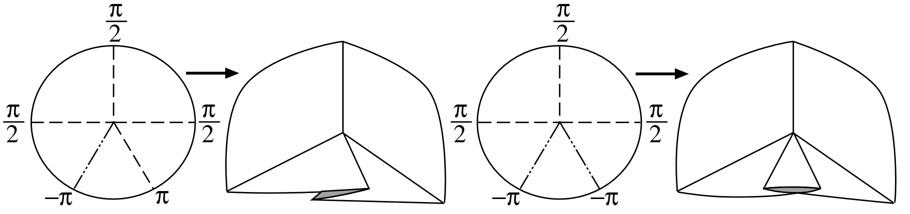

Fig. 4. How flat flaps can also self-intersect in a non-flat fold.

the completed fold under the folding map f and check to see if they intersect, or if any faces intersect any edges... but how? While all of these are characterizations of sufficiency for single-vertex folds, none of them appear to be testable (yet). This is certainly a problem which merits further investigation.

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