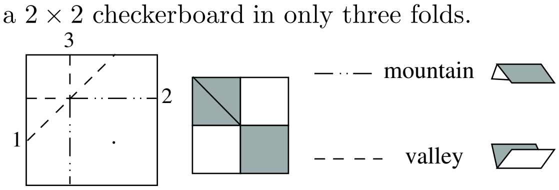
Origami Quiz

by Thomas Hull

Paper is all around us. Every day we fold paper. So test your knowledge of and your ability to explore this simple, everyday activity.

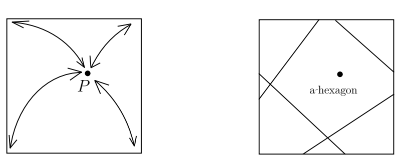
1. Find a square piece of paper that is white on one side and colored on the other. From such paper it is possible to use the contrasting colors to fold any $n \times n$ checkerboard. Trying to do this in as few actual folds as possible can be a perplexing challenge. Below is shown how to fold a 2×2 checkerboard in only three folds.



Observant skeptics may quarrel with the fact that we divided the paper into thirds "for free" in the above solution. However, all we are counting are actual folds used in the end, and we adopt the convention that any landmarks needed (like a one thirds mark) can be made beforehand without counting to our fold total.

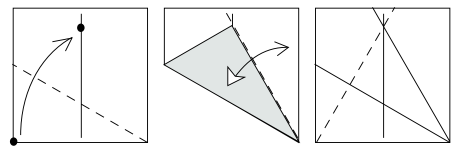
How would one fold a 3×3 checkerboard? What is the fewest number of folds needed?

2.a. (By Kazuo Haga, [2]) Take a square piece of paper and let P be any point on the square. Taking one at a time, fold and unfold each corner of the square to the point P. When you're done, P should be contained in some polygon determined by the creases and, possibly, the sides of the square. How many sides can this polygon have? Which regions of the paper give which polygons?

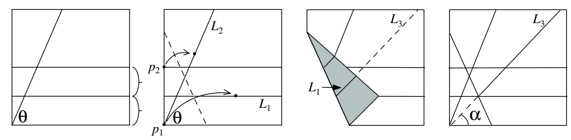


2.b. Does it make sense to consider the point P to be chosen *outside* the square? What if we instead used a rectangle?

3. The below diagram shows how one can fold an equilateral triangle in a square piece of paper. Does it work? Is this the largest equilateral triangle that can me made from a square?



4. What interesting thing is the below folding procedure doing to the angle θ? (Hint: what is the angle α?)



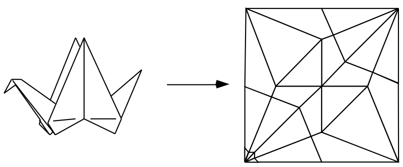
5. We are all familiar with geometric constructions using a straightedge and compass. As early at the nineteenth century geometers have also been using paper folding as a geometric construction medium. What are the basic folds (operations) that define paper folding? For example, one clear fundamental fold is, "Given two points p_1 and p_2 we can make a crease line that passes through p_1 and p_2." Think of another one that allows us to construct angle bisectors via folding. Try to make your list as complete as possible. (The "moves" in Problem 3 and 4 above should be represented, for example.)

6. When you did Problem 5 and considered the fold in Problem 3 you probably included something like the following into your list of basic folds: *Given two points p_1 and p_2 and a line L_1, we can sometimes make a crease that passes through p_2 and places p_1 onto L_1.* Why do we need to say "sometimes"? What conditions on p_1, p_2 and L_1 will make it always work?

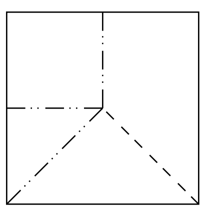
7. If we think of the paper as lying in $\mathbb{R}^2$ (or if you prefer, $\mathbb{C}$), what type of algebraic equation is the basic folding operation in Problem 6 solving?

8. If we consider the piece of paper to exist in the complex plane, define *origami numbers* to be those points in $\mathbb{C}$ that are constructible via paper folding. How does the field of origami numbers compare to the field of numbers constructible by straightedge and compass when we consider the answer to Problem 7? What does Problem 4 tell us?

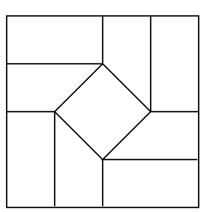
9. Most models that you'll find in origami books are *flat* models. That is, when completed they can be pressed in a book without introducing new creases. The classic flapping bird (shown below) is one example. Take any flat origami model, unfold it, and consider the crease lines used in the final folded form (i.e., we are not considering auxiliary creases made during the folding process but not used in the end). How many colors does it take to color the regions in between the creases in the crease pattern, making sure that no two neighboring regions (sharing a boundary line) receive the same color?



10.a. Creases come in two types: *mountains*, which are convex, and *valleys*, which are concave. These are often distinguished in origami instructions by different types of dashed lines (see the figure in Problem 1 above). But a paper folder cannot just chose which creases will be mountains and which will be valleys willy-nilly! Indeed, the below single-vertex crease pattern can fold flat, but not using the prescribed mountain-valley choices. Why is this impossible to fold flat?

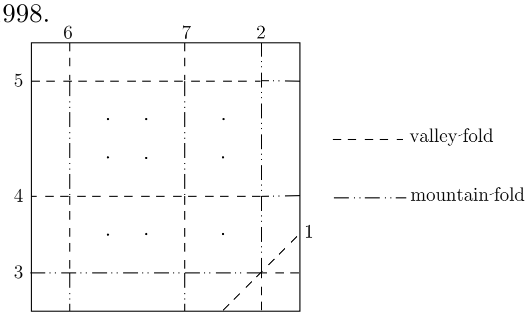


10.b. The above vertex can be arranged, or tessellated, with copies of itself four times to make a very interesting crease pattern, shown below, called the *square twist*. Use what you deduced from Part a to compute how many valid mountain-valley assignments exist for this crease pattern.

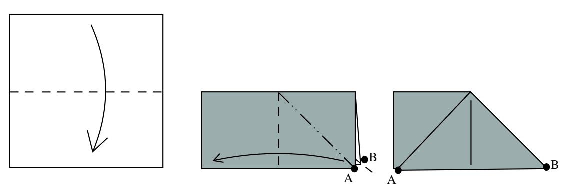


Solutions (of a sort) to the Origami Quiz

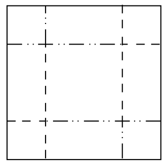
1. The 3×3 checkerboard can be folded in only 7 folds. The below solution is due to Kozy Kitajima, and was presented at the Gathering for Martin Gardner conference (Atlanta, GA) in 1998.



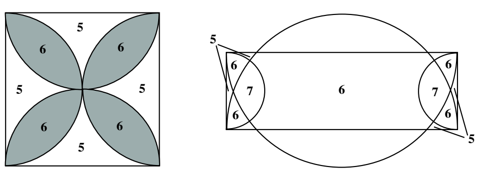
Readers will be tempted to generalize this puzzle to $n \times n$ checkerboards, but it quickly becomes extraordinarily difficult. For the 4×4 case, the best solution known to the author requires 14 folds, and this assumes that we allow an origami move known as a "squash fold" to count as one fold. (A squash fold is shown in the below middle figure.)



In fact, the question of what "counts" as one fold is very non-trivial. Bitter debates on this very question emerge when practiced origamists face this puzzle. For example, the below crease pattern presents a "1-fold" solution to the 2×2 checkerboard puzzle. That is, if each crease is carefully made beforehand, then all of the creases (in their proper mountain/valley directions) must be folded simultaneously to obtain the checkerboard pattern. Since only one motion is required, does this count as one fold? (This is tricky to do; readers are encouraged to try it persistently. And the reward is great since it actually makes a 2×2 checkerboard on *both* sides of the paper. This is an example of what origamists call an *iso-area model*, where both sides of the paper are doing the same thing, up to rotation and reversal of the creases. Origamist Jeremy Shafer has a similar, iso-area, "1-fold" solution to the 4×4 checkerboard puzzle.)



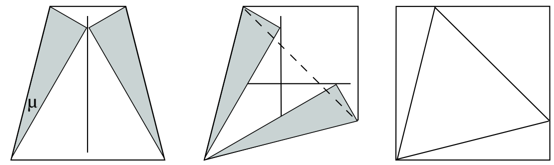
2. Imagine our piece of paper is the plane, $\mathbb{R}^2$, and our square is drawn on the plane with vertices at $(\pm 1, \pm 1)$. If we ignore the square, it is clear that if we fold and unfold the vertices to our random point P, the crease lines will form four sides of a quadrilateral containing P. Then answering Part a reduces to determining how the sides of the square intersect this quadrilateral. If P is located at one of the square's vertices or at $(0, 0)$, then P will be contained in a square on the paper. Otherwise if P is close enough to one of the sides of the square, then that side will cut across the quadrilateral made by the folding. We can determine when this will happen by drawing semicircles of radius 1 centered at the midpoints of each side of the square. How these semicircles overlap determines the solution, shown below left.



As for Part b, thinking of the paper as being the infinite plane allows us to consider P to be chosen outside the square. However, after we make our folds P will be located in an infinite region, and it is an interesting game to think of how we can redefine what we choose to "count" as our polygon in such cases. In any case, the differences will be determined by extending our semicircles in Part a's solution to full circles.

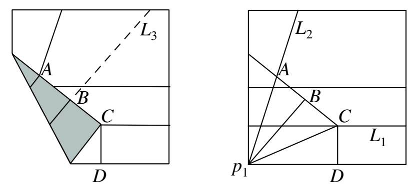
Rectangular paper is handled in the same way as in part a. Interestingly, heptagons can be produced. (See the above right illustration.)

3. If we let the side of the square be of length one, then the triangle made by this folding procedure is equilateral because its sides all have length one. (Its left and right sides are both images of the bottom side under folding.) It is not the biggest equilateral triangle possible, however. The biggest is symmetric about a diagonal of the square, and a folding method for such a maximal triangle is shown below. (Note that the angle μ equals 15°. This "proof without words" construction was devised by Emily Gingrass, Merrimack College class of 2002.)



Actually proving that this is the equilateral triangle of maximal area that can be inscribed in a square is a fun trig/elementary calculus problem.

4. This is an origami method of trisecting an angle. Drawing some auxiliary lines and unfolding the paper can prove the trisection works. In the below picture, argue that the segments $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$ are all of the same length.



5. Lists of basic origami operations may vary. A lot depends on how one sets things up, and we do not want our list to be redundant. But an initial list of folding operations might look something like the following:

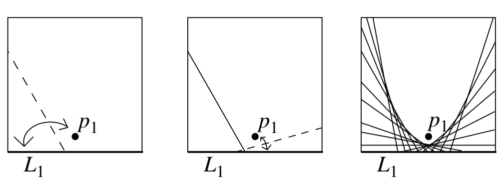
1. Given two points p_1 and p_2, we can make a crease line connecting them.
2. Given two points p_1 and p_2, we can fold p_1 onto p_2. (This creates the perpendicular bisector to line segment $\overline{p_1 p_2}$.)
3. Given two lines, L_1 and L_2, we can fold line L_1 onto L_2. (Angle bisectors.)
4. We can locate points where two non-parallel lines intersect.
5. Given a line L and a point p not on L, we can make a fold through p that is perpendicular to L, in other words, folding L back onto itself so that the crease passes through p. (Dropping a perpendicular.)
6. Given two points p_1 and p_2 and a line L_1, we can, whenever possible, fold p_1 onto line L_1 so that the resulting crease passes through point p_2. (This was part of the construction in Problem 3, where p_2 was one corner of the paper.)
7. Given a point p_1 and two lines L_1 and L_2, we can make a crease placing p_1 onto L_1 that is perpendicular to L_2.
8. Given two points p_1 and p_2 and two lines L_1 and L_2, we can, whenever possible, make a crease that simultaneously places p_1 onto L_1

and p_2 onto L_2.

Operations 1-3, 5, 6, and 8 were formulated by Humiaki Huzita. (It is not certain if he was the first to do this, but he was the first to publish them. See [5] and [6].) Move 7 is, amazingly enough, a very recent addition developed by Koshiro Hatori (see [3]). Most readers will not have thought of operation 8, although it does appear in Problem 4. Also recently, Robert Lang ([7]) has proven that these operations exhaust all that origami can do. He does this by beginning with the premise that all we can do in origami is fold points and lines to each other, and runs through all the possibilities while formalizing the degrees of freedom one has when folding one object to another.

Also, Koshiro Hatori claims that most of these operations can be thought of as special cases of operation 8. Can you find a way to make this work?

6. The basic origami operation cited in this problem (number 6 in the above list) cannot be performed if the point p_2 is poorly positioned with respect to p_1 and L_1. To see what is going on, do the following exercise: Take a piece of paper and let the bottom side be line L_1 and take a random point p_1 on the paper. Fold and unfold p_1 onto L_1 at many different places, making a sharp crease every time you do so. The below figure illustrates what you should see.



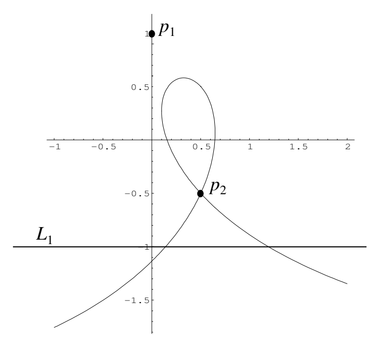
This exercise makes one suspect that the process of folding a point p_1 to a line L_1 is actually creating a crease line that is tangent to the parabola whose focus is p_1 and directrix is L_1. There are a number of ways to prove this; for an analytic approach let $p_1 = (0, 1)$, let L_1 be the x-axis, and find the equation of the crease that results when p_1 is folded to an arbitrary point $(t, 0)$ on L_1. Then take the envelope of this family of lines – a parabola should result.

Now, this situation is at play in folding operation 6, and clearly if the point p_2 is chosen to be in the interior of the convex hull of the parabola with focus p_1 and directrix L_1, then the operation will be impossible to perform.

7. See the solution to Problem 6. Since we get a parabola, folding operation 6 is actually solving a quadratic equation for us. (Can you give an explicit method of solving $ax^2 + bx + c = 0$ where a, b, and c are positive integers?)

8. The set of numbers constructible with straightedge and compass is the smallest subfield of $\mathbb{C}$ that is closed under taking square roots. So straightedge and compass can solve quadratic equations, but certainly cannot construct any algebraic $\alpha \in \mathbb{C}$ whose minimal polynomial is cubic. The classic proof that a straightedge and compass cannot trisect an angle, for example, is built on $\cos 20°$ being degree 3 over the rationals.

In Problems 6 and 7 we saw that the origami operation 6 proves that paper folding can solve quadratic equations. Thus the set of origami numbers contains the set of straightedge and compass constructible numbers. Furthermore, since we know that paper folding can trisect angles, we know that the field of origami numbers strictly contains the field of straightedge and compass numbers. Actually, folding operation 8 turns out to allow us to solve general cubic equations. As evidence, the below figure dipicts the locus of possible images of $p_2 = (.5, -.5)$ as $p_1 = (0, 1)$ is folded repeatedly onto line L_1 which is $y = -1$. This graph certainly looks cubic, and deriving its equation can be done using similar analytic methods to those in the solution to Problem 6. For more information, see [1].



9. All flat origami crease patterns are 2-face colorable. The proof is simple: take your flat-folded origami model and lay it on a table. Color all regions of the paper yellow if they face up, away from the table and all regions pink if they face down, into the table. Any two neighboring regions of the crease pattern will have a crease line in between them, and thus point in different directions when folded, insuring that they receive different colors. Thus this is a proper 2-face coloring of the crease pattern.

One can also prove this using only graph theory. First argue that all vertices in the interior of the paper of a flat model have even degree. Thus if we consider the crease pattern to be a graph, where the boundary of the square also contributes edges to the graph, the only odd degree vertices would possibly be on the paper's boundary. Create a new vertex, v in the "outside face" and draw edges from it to all the odd degree vertices on the paper's boundary. Graphs always have an even number of vertices of odd degree, so the degree of v is even, and the new graph we've created has all vertices of even degree. It is an elementary graph theory fact that all such graphs are 2-face colorable (prove that its dual is bipartite), and removing the vertex v then gives a 2-face coloring of the original crease pattern.

10.a. The problem is with the two mountain creases that surround the 45° angle at this vertex. The two angles neighboring the 45° angle are both 90°. Thus, if the creases surrounding the 45° angle have the same mountain-valley (MV) parity, then the two 90° angles will both be forced to cover up the 45° on the same side of the paper. If these creases are then pressed flat, the two large angles will be forced to intersect one another, and self-intersections of the paper are not allowed (unless one is folding in the fourth dimension, which we assume we are not!).

It turns out that mountains and valleys can be assigned to these crease lines and be flat-foldable if (1) the creases surrounding the 45° angle are not the same and (2) the number of mountains and the number of valleys differ by 2. (This last result holds for general flat vertices and is known as Maekawa's Theorem. See [4] for more information.)

b The answer is 16. There are several ways to enumerate the valid MV assignments. One way is to look at the inner "diamond" whose MV assignment will force the MV parity of the rest of the creases. (Why?) The inner diamond creases can have any combination of mountains or valleys, giving $2^4 = 16$ possibilities.

References

[1] R.C. Alperin, *A mathematical theory of origami constructions and numbers*, New York Journal of Mathematics, Vol. 6, (2000) 119-133 (available online at http://nyjm.albany.edu).

[2] K. Haga, Fold paper and enjoy math: origamics, *Origami^3: Proceedings of the Third International Meeting of Origami Science, Mathematics, and Education*, T. Hull ed., A.K. Peters, (2002) 307-328.

[3] K. Hatori, Origami versus straight edge and compass, http://www.jade.dti.ne.jp/ hatori/library/conste.html

[4] T. Hull, The combinatorics of flat folds: a survey, *Origami^3: Proceedings of the Third International Meeting of Origami Science, Mathematics, and Education*, T. Hull ed., A.K. Peters, (2002) 29-38.

[5] H. Huzita, "Understanding Geometry Through Origami Axioms: is it the most adequate method for blind children?" in the *Proceedings of the First International Conference on Origami in Education and Therapy*, J. Smith ed., British Origami Society, 1992, pp. 37-70.

[6] H. Huzita and B. Scimemi, "The Algebra of Paper-folding (Origami)", in the *Proceedings of the First International Meeting of Origami Science and Technology*, H. Huzita ed., 1989, pp. 205-222.

[7] R. Lang, personal communication.