# Unit Origami as Graph Theory

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*ABSTRACT: In the search for innovative material to teach high school mathematics students more and more teachers are turning to graph theory. This subject provides an absract, powerful and very modern way to model any type of network structure, from telephone networks to the human central nervous system. One of the most engaging ways to teach graph theory is with unit origami. In this paper we will discuss examples and techniques for doing just this.*

## 1. Introduction

Unit origami has its roots in complex origami figure design. If a designer could not create the desired object with one piece of paper, several sheets might make the task easier. However, thanks mainly to the popular works of T. Fuse and K. Kasahara ([4], [5]), people nowadays think of modular origami as using several, sometimes hundreds of pieces of paper to create various geometric forms. The charm of such origami is that simple, easy to fold modules can lead to very complex, intricate structures. Also, since more than one piece of paper is involved, one can experiment with the multitude of color patterns that are possible in any one modular origami work.

In exploring such geometric structures and color patterns, a good understanding of 3-dimensional polyhedral geometry is needed. In this paper, we will discuss how such a geometric understanding can be reached via a branch of mathematics called *planar graph theory*[2]. By examining specific examples we will show how studying planar graphs can lead to the generation of more complex modular origami models and touch upon the inverse direction as well: how studying modular origami can lead to a better understanding of planar graph theory. Such a correlation would certainly be valuable to educators interested in implementing more graph theory into their curriculum. The author himself has found unit origami not only to be a great educational device, but also a great source of ideas and inspiration in his own mathematical research on planar graphs.

## 2. What is a planar graph?

A **graph** is what mathematicians use to model networks. It consists of a collection of points, called **vertices**, which are connected by lines, called **edges**. Intuitively, graphs can be thought of as simply a bunch of vertices connected by edges, but for completeness a technical definition follows:

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[2]For a more rigorous introduction to graph theory, see [2] or [3].

**Definition:** A **graph** $G$ is pair $(V, E)$, where $V = \{v_1, v_2, ..., v_n\}$ is a set of vertices and $E \subset V \times V$ is a set of edges.

Thus each edge $e \in E$ is of the form $e = (v_i, v_j)$ and this means that the vertices $v_i$ and $v_j$ are connected by an edge. Figure 2.1 shows examples of some graphs.

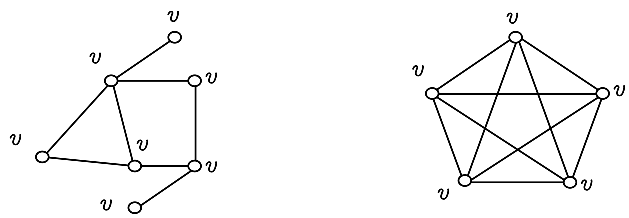


Figure 2.1: Some graphs.

A **planar graph** is a graph that can be drawn so that the edges intersect only at the vertex points. In Figure 2.1, the left graph is planar, while the right graph is not. You might need to experiment with a pencil and paper to convince yourself of this fact! There are many different ways in which we could draw the right-most graph, but all of them require edges to overlap.

Our interest is in planar graphs because they provide a natural way of studying polyhedra. Imagine a polyhedra (say, a cube) in 3-dimensional space. Surely it too can be thought of as simply a collection of vertices (the corners) connected by edges! Furthermore, we could take one side (a **face**) of this polyhedra and stretch it out, as if the edges were made of elastic, so as to *embed* the polyhedra in a 2-dimensional plane. Doing this turns the polyhedra into a planar graph, and any polyhedra can be embedded in the plane in this way. To get a better understanding of this, see Figure 2.2, where planar graph embeddings of the five Platonic solids are shown. (The cube might be the easiest to visualize.)

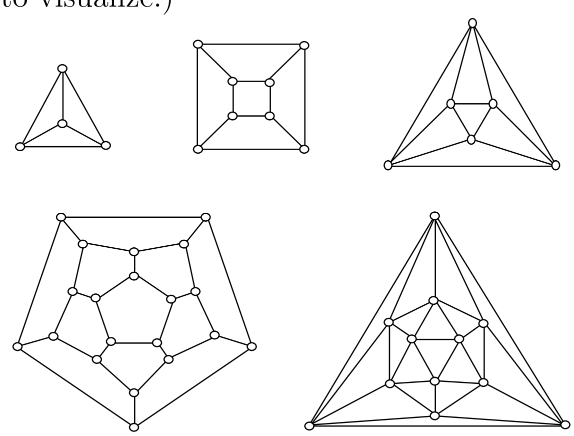

Figure 2.2: The Platonic solids in planar graph form.

The study of planar graphs has a long history in mathematics, going back to the time of Euler (1736). As a result, there are many known facts about planar graphs. For example, Euler himself proved that if $v$ is the number of vertices, $e$ the number of edges, and $f$ the number of faces in a planar graph, then

$$v - e + f = 2.$$

This is known as *Euler's formula*, and since it applies to planar graphs, it also applies to polyhedra.

Thus since polyhedra and the Platonic solids are already a common subject for the high school geometry classroom, this link between graph theory and polyhedra could provide a natural starting point for an introduction to graph theory.

But let us now turn our attention to specific examples of how planar graph knowledge can help us design modular origami structures. In the following we will assume a basic familiarity with unit origami. That is, even though the topics, like creating a spikey module, may seem advanced origami-wise, they are merely provided to illustrate the powerful link between graph theory and unit origami. As you read it may help to think of how these subjects might be introduced into a high school mathematics classroom. And remember that origami, with it's bright colors and amazing shapes almost needs no introduction!

## 3. "Spiked" polyhedra modules.

Many modular folds create versions of "stellated", or "spiked" polyhedra. That is, the form they produce is a polyhedra with pyramids "capped" on each face. The result is usually a spikey-looking object. Some examples can be found in [5].

After designing such a module that could produce spiked versions of almost any polyhedra, the author wondered what was the fewest number of colors needed to "color" such an object. By a **coloring** we mean an assignment of colors to the faces of the spiked polyhedra such that no two adjacent faces have the same color. Specifically, this would be called a **face-coloring** of the polyhedra.

This question can be answered rather easily using a specific case of a theorem from graph theory called Brook's Theorem. To understand it, we must define the **degree** of a vertex in a graph to be number of edges coming out of that vertex.

**Brook's Theorem (cubic case):** *A graph is vertex-colorable in 3 colors if its maximum vertex degree is 3, unless it is the tetrahedron.*

Note that this is a theorem about *vertex-colorings*, i.e., an assignment of colors to the vertices of a graph such that no two vertices connected by an edge have the same color. Thus, one might ask, "How does this relate to face-coloring spiked polyhedra?"

Suppose we have a polyhedra which we wish to spike (i.e., place pyramids on each of its faces). If we think of this polyhedra as a planar graph, then the spiking process amounts to placing a *new* vertex in each of the polyhedra's faces, and then adding edges connecting the vertices along the face to the new vertex (see Figure 3.1).

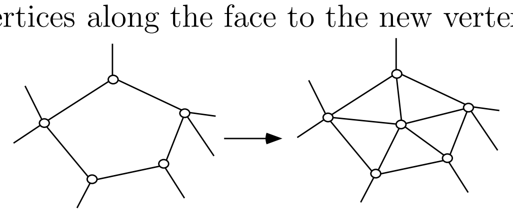

Figure 3.1: The spiking of a face.

In fact, if $G$ is a planar graph, let us define $S(G)$ to be $G$ "spiked". That is, $S(G)$ is the new graph obtained by doing the capping operation shown in Figure 3.1 to every face of $G$. Then notice that every face of $S(G)$ will be a triangle, and this is true for any planar graph (polyhedra) that we start with.

Now we wish to color the faces of $S(G)$. Doing this would be equivalent to coloring the *vertices* of the **dual** of $S(G)$. The dual of $S(G)$, which is denoted $S(G)^*$, is the new graph obtained by reversing the roles of the vertices and faces of $S(G)$. Vertices of $S(G)$ will be faces of $S(G)^*$, and faces of $S(G)$ will be vertices of $S(G)^*$. In technical terms, $S(G)^*$ is the graph whose vertex set is the collection of faces of $S(G)$, where two vertices of $S(G)^*$ are connected by an edge if and only if the corresponding faces in $S(G)$ are adjacent (next to each other). Figure 3.2 demonstrates how the dual of the cube is the octahedron.

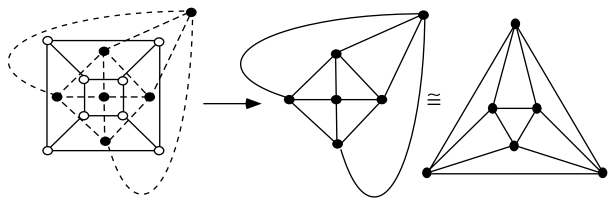

Figure 3.2: (Cube)$^*$ =Octahedron.

But what kind of graph is $S(G)^*$? Remember that $S(G)$ was $G$ "spiked". But what do the faces of $S(G)$ look like? Well, the spiking process covers each face of $G$ with a spike, and the sides of each spike are triangles. Thus each face of $S(G)$ is a triangle! Since each *face* of $S(G)$ is a triangle, each *vertex* of $S(G)^*$ (the dual) will be of degree three! So the maximum vertex degree of $S(G)^*$ is three, and Brook's Theorem tells us that $S(G)^*$ is 3 vertex-colorable. Since each vertex of $S(G)^*$ corresponds to a face of $S(G)$, this proves that $S(G)$ can be face-colored using only three colors.

This may seem very technical, but the punchline is that *any structure made from unit origami that produces spiked pokyhedra can be made using only three colors.* But coming up with this three coloring may not be easy! It can be quite a challenge for students (and even adults!) to make an origami spiked icosahedron using only three colors so that no two pieces of paper with the same color touch. Then such explorations can gradually lead to the above discussion as to *why* this can be done.

Another interesting fact is that the structure of $S(G)$ can be understood very well, no matter what polyhedron $G$ is. Recall what it means to *truncate* a polyhedron, i.e., to chop off each corner, creating a new face where the corner used to be. In terms of a planar graph, this is equivalent to replacing each vertex of degree $n$ with an $n$-gon, as illustrated in Figure 3.3. In fact, let us define $T(G)$ to be the graph obtained by truncating all the vertices of $G$.

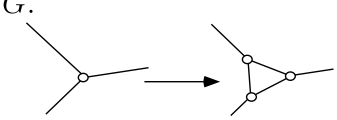

Figure 3.3: The truncation of a vertex.

It is left as an exercise to the reader to prove that if $G$ is the planar graph of a polyhedra, then

$$S(G)^* \cong T(G^*).$$

That is, the dual of $G$ spiked is congruent (the same as) the truncation of the dual of $G$. This handy little fact means that to face-color a spiked cube, for example, one could equivalently vertex-color a truncated octahedron.

## 4. Cubic skeleton modules.

In the above section we saw how spiked polyhedra are related to graphs in which each vertex is of degree 3. A graph with only degree 3 vertices is called a **cubic** graph. Cubic graphs play a central role in planar graph theory (as we will see below), and it is interesting that many modular origami units are "cubic" as well. That is, we call a modular unit **cubic** if it takes three units to form a corner, or vertex, of the underlying polyhedra. The Sonobé unit (see [6]) is one example of a cubic unit.

One large class of cubic modules are what Jeannine Mosely refers to as "zig-zag" modules. They involve accordion-pleating a square into thirds or fourths to create a strip. This strip then has "pockets" on both of its long sides. When the strip is folded to produce flaps for these pockets, any one of a number of different modules can be the result. Three examples which can be folded from a square accordian-pleated into fourths are shown in Figure 4.1.

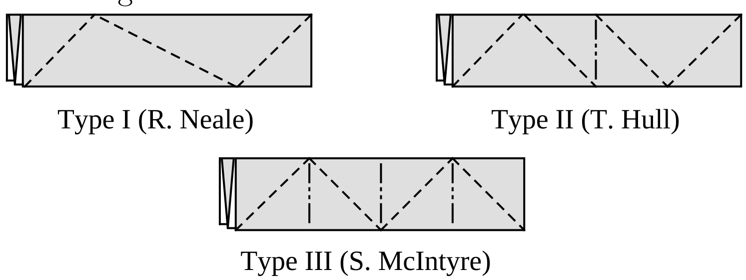


Figure 4.1: Three zig-zag modules.

These modules can be used to produce "skeletons" of polyhedra, i.e., the units play the role of the polyhedra's *edges*. Neale's type I module is the most inflexible of the three. When three are interlocked, they force a rigid angle which can produce the skeleton only of a dodecahedron. Types II and III offer more variety. Type II can generate skeletons of any polyhedra with neither triangular nor square faces. McIntyre's type III unit produces only polyhedra in which every face has an even number of sides.

One reason such modules are of interest is because they offer us a fun and easy way to explore **edge-colorings** of cubic planar graphs. An edge-coloring is similar to vertex and face-colorings; one colors the edges of the graph so that no edges which meet at a vertex are of the same color. In a cubic planar graph, each vertex has three edges coming out of it, so one might hope that we could always get away with using only three colors to properly color a cubic modular structure.

In fact, we can always get away with three colors, but it is by no means easy to see why. This problem, whether or not we can color the edges of a cubic planar graph with only three colors, turns out to be equivalent to the famous Four Color Theorem, a theorem with one of the most notorious histories in mathematics![1]

[1]The Four Color Theorem states that any planar graph can be vertex-colored using only 4 colors. It was an open conjecture for a hundred years, until Appel and Haken proved it in 1977 by breaking

Thus the possibility exists that by 3-coloring cubic modular works, further light might be thrown on the search for a short proof of the Four Color Theorem. How this could possibly happen is another matter. But consider the exercise of 3-coloring the edges of a dodecahedron, which can easily be realized using unit type I in Figure 4.1. When this done, the observant folder might notice that no matter how you color it, the pentagonal faces which are opposite each other always have the same color pattern! This fact is not easily noticed when coloring the edges of the dodecahedron on paper.

Moving from edge-coloring to planar graph structure, consider the unit type II in Figure 4.1. This unit locks together quite well, and made the author wonder how large a structure could be made from such a unit. Of course, a dodecahedron could be made with 30 units, and, remembering that this unit can produce any polyhedron without triangle or square faces, a soccer ball (truncated icosahedron) can be made from 90 units. But what else?

There turns out to be a very easy way to start with the dodecahedron and generate an infinite class of polyhedra with only pentagons and hexagons as faces. Let $D$ be the planar graph of the dodecahedron. Then the dual $D^*$ is the icosahedron (see if you can convince yourself of this!), and thus $T(D^*)$ is the soccer ball.

But look at what this process did. Each face of $D$ is a pentagon, so each vertex of $D^*$ has degree 5. When this degree 5 vertex is truncated, it will produce a pentagonal face! Thus the operation

$$D \to T(D^*)$$

preserves pentagonal faces (see Figure 4.2). But where did the hexagons present in $T(D^*)$, the soccer ball, come from? Well, each vertex of $D$ has degree 3, which turns into a triangular face in $D^*$. In $T(D^*)$ every vertex of this triangle has been truncated, resulting in a hexagon face! Thus when we peform $D \to T(D^*)$, each pentagon face remains a pentagon and each vertex becomes a hexagon.

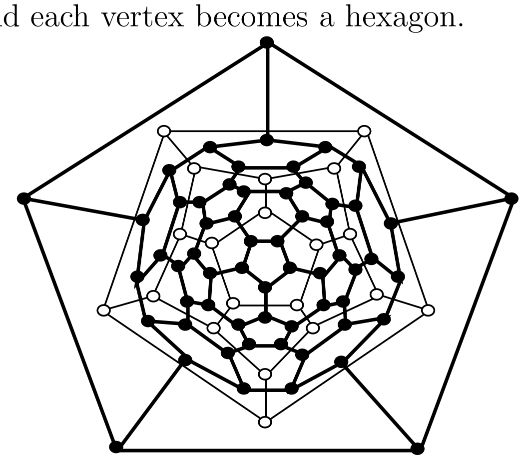

Figure 4.2: The relation between $D$ and $T(D^*)$.

Then what will $T(T(D^*)^*)$ be? Each pentagon of $T(D^*)$ will remain a pentagon, and a similar argument shows that the hexagons of $T(D^*)$ will remain hexagons too.

the problem down into 1936 cases and checking each of these *by computer*. Thus the Four Color Theorem was the first "computer proof." See [8] and [9] for more information

But each vertex (still of degree 3) of $T(D^*)$ will create a new hexagon in $T(T(D^*)^*)$, thus producing a new, bigger polyhedra still having only pentagons and hexagons as faces. This process can be continued to produce an infinite family of such polyhedra.

It is a simple exercise to show that if $G$ has $n$ edges, then $T(G^*)$ has $3n$ edges. Thus, the number of type II units needed to create any of these structures is easy to calculate. The first four are given in the below table.

| | |
|---|---|
| $D$ | 30 units |
| $T(D^*)$ | 90 units |
| $T(T(D^*)^*)$ | 270 units |
| $T(T(T(D^*)^*)^*)$ | 810 units |

The author has constructed the third iteration sphere (810 units) from 2-inch and 3-inch paper. They hold together quite well.

But things get even more crazy! Suppose we restricted outselves to *only* pentagon and hexagon faces. That is, what kinds of polyhedra can we make with the type II unit (or any other cubic unit) that have only pentagons and hexagons for faces. (No triangles, squares, septagons, etc.) It turns out that *every* such polyhedra **must** have **exactly** 12 pentagon faces. No more, no less. Why is this true? We can see why using a little more graph theory. Indeed, let's state this as a Theorem and try to prove it:

**Theorem:** *Any structure made from cubic units with only pentagon and hexagon faces will have exactly 12 pentagon faces.*

**Proof:** We want to show this for an arbitrary polyhedra. So suppose we have a polyhedra that satisfies our hypotheses. (I.e., it's cubic and has only pentagon and hexagon faces.) We want to use some graph theory, so think of this polyhedra as a graph $G$.

Let $v=$ the number of vertices in $G$.

Let $e=$ the number of edges in $G$.

Let $f=$ the number of faces in $G$.

Then we know Euler's formula must hold, that is, $v - e + f = 2$.

Now think about the fact that $G$ is cubic. Imagine that we tried to count all the edges in $G$. One way to do this would be to look at every vertex and see that there are exactly three edges coming out of each vertex. So for every vertex we have three edges, but if we count the edges in this way we'll count each edge *twice*! This is because each edge connects two vertices. So each vertex gives us 3 edges, and this counts each edge twice. In other words, $3v = 2e$. We can then combine this with Euler's formula to get a new formula:

$$3v = 2e \Rightarrow v = \frac{2}{3}e \Rightarrow \frac{2}{3}e - e + f = 2 \Rightarrow -\frac{1}{3}e = -f + 2$$
$$\Rightarrow e = 3f - 6 \qquad (*)$$

OK. We'll use equation (*) in a minute. But now consider the fact that there are only hexagon and pentagon faces in $G$. Let let $f_5=$ the number of pentagon faces in $G$ and let $f_6=$ the number of hexagons. (Then our goal is to show that $f_5 = 12$, right?)

Then the fact that the pentagons and hexagons make up all the faces of $G$ tells us that

$$f_5 + f_6 = f \qquad (**)$$

We'll use this formula in a minute also. But now let's count the edges again! Sure, but this time we'll count them from the faces. Each pentagon face has 5 edges surrounding it, and each hexagon has 6 edges around it. If we count the edges this way we get $5f_5 + 6f_6 = 2e$ since again we count each edge twice when we do this. (Each edge separates exactly two faces.) We can then play with this formula:

$$5f_5 + 6f_6 = 2e \Rightarrow 5(f_5 + f_6) + f_6 = 2e \Rightarrow 5f + f_6 = 2e$$

$$\text{Then use (*): } 5f + f_6 = 6f - 12 \Rightarrow f_6 = f - 12$$

$$\text{Then use (**): } f = f_5 + f - 12 \Rightarrow f_5 = 12.$$

Therefore the number of pentagon faces is exactly 12. Wow!

**5. Conclusion?**

We hope the reader is convinced by these examples that knowledge of planar graph theory can lead to a deeper understanding of polyhedral structure, and thus a deeper understanding of the potentials of modular origami. In fact, the author's experience has been that only after studying the links between planar graphs and modular origami was he able to start designing modular units of his own. Also, we have only touched upon a few examples here. The reader is encouraged to explore the "hidden graph" structure underlying all modular origami works.

Furthermore, we hope that the reader sees the potential of what one can learn by playing with modular origami. Making unit origami polyhedra seems innocent enough at first, but there's a wealth of ideas and mathematical structure lurking underneath. Because of this unit origami can be a wonderful vehicle for exposing students to mathematical explorations. For the author it has proven to be a great tool for teaching high school, and even college students to look for the "math" in the world around them.

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