MA 343 Real Analysis **11/20/01**

More Details on the Riemann Integral

OK. Let's define the Riemann integral in more thorough detail than they do in the Bressoud book.

Let $f(x)$ be a bounded function defined on an interval $[a, b]$, and let $P = \{x_0 = a, x_1, x_2, ..., x_n = b\}$ be a partition of $[a, b]$. Define for $i = 1, ..., n$

$$\begin{aligned} M_i &= \text{lub}\{f(x) \; : \; x \in [x_{i-1}, x_i]\} \\ m_i &= \text{glb}\{f(x) \; : \; x \in [x_{i-1}, x_i]\} \end{aligned}$$

Note that lub and glb will just act like max and min, respectively, for most functions we'll see.

This notation allows us to define the upper and lower Riemann sums, which you may remember from Calculus I. We let $\Delta x_i = x_i - x_{i-1}$. Then the **upper sum** of f with respect to P is

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

and the **lower sum** of f with respect to P is

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Exercise 1: Find $U(f, P)$ and $L(f, P)$ for $f(x) = e^x$ on the interval $[0, 2]$ with the partition $P = \{0, 1/2, 1, 3/2, 2\}$.

Now, these upper and lower sums are NOT the actual value of the integral. You have to take finer and finer partitions to get that, and if it works then the upper and lower sums should become **equal** in the limit. Here's how we say this mathematically...

Define the **upper integral** of $f(x)$ on $[a, b]$ to be

$$U(f) = \text{lub}_{\text{all partitions } P \text{ of } [a,b]}\{U(f, P)\}$$

and define the **lower integral** of $f(x)$ on $[a, b]$ to be

$$L(f) = \text{glb}_{\text{all partitions } P \text{ of } [a,b]}\{L(f, P)\}$$

If $U(f) = L(f)$ then we say that $f(x)$ is **Riemann intregrable** and we denote

$$\int_a^b f(x)dx = U(f) = L(f).$$

Exercise 2: Show that $f(x) = x + 1$ is Riemann integrable on the interval $[0, 1]$ by using the partition $P_n = \{0, 1/n, 2/n, 3/n, ...(n-1)/n, 1\}$. That is...

(a) Compute $L(f, P_n)$.

(b) Compute $U(f, P_n)$.

(c) Show that $L(f) \geq \lim_{n\to\infty} L(f, P_n) = 3/2$ and that $U(f) \leq \lim_{n\to\infty} U(f, P_n) = 3/2$. Then show that $U(f) = L(f)$.

Exercise 3: The Riemann integral can handle discontinuous functions too (sometimes)! Show that the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

is Riemann integrable over the interval $[-1, 1]$.

Exercise 4: Show that the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

defined on the interval $[0, 1]$ is NOT Riemann integrable! Hoo-boy!