Mathematical Methods in Origami

University of Tokyo, Komaba
Day 1 Slides, Dec. 16, 2015
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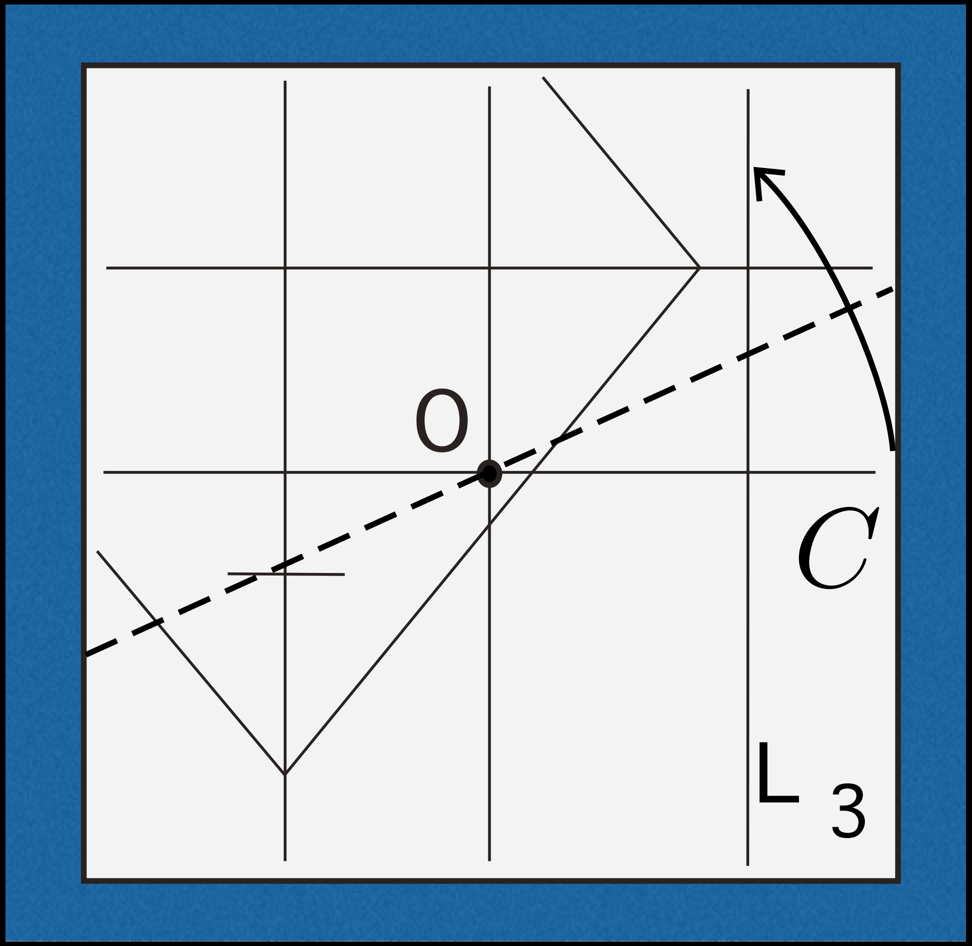
Lecture notes and assignments available at:
http://mars.wne.edu/~thull/tokyo/class2015.html

Overview of Origami-Math

- origami geometric constructions



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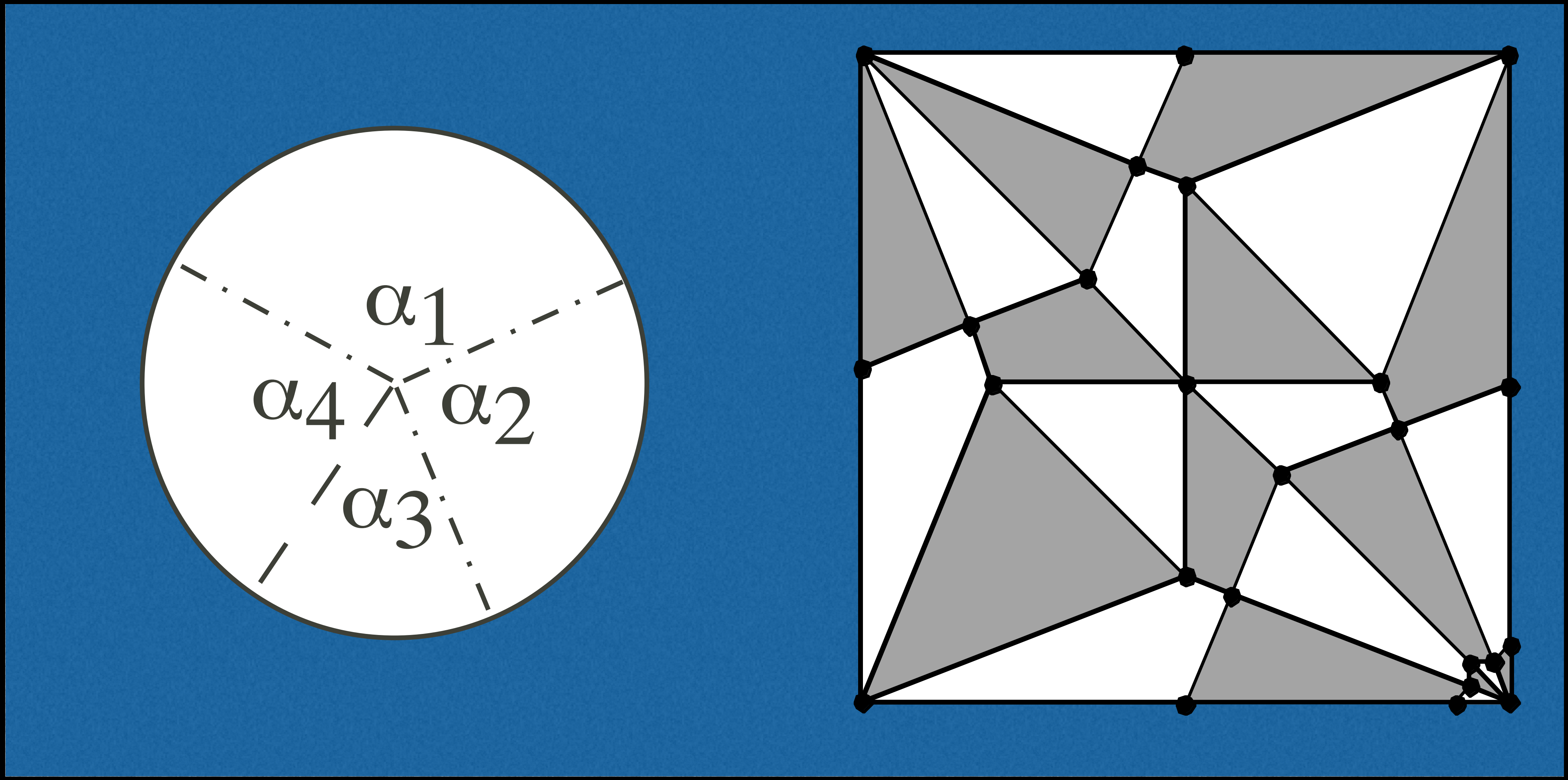
- combinatorial geometry of origami



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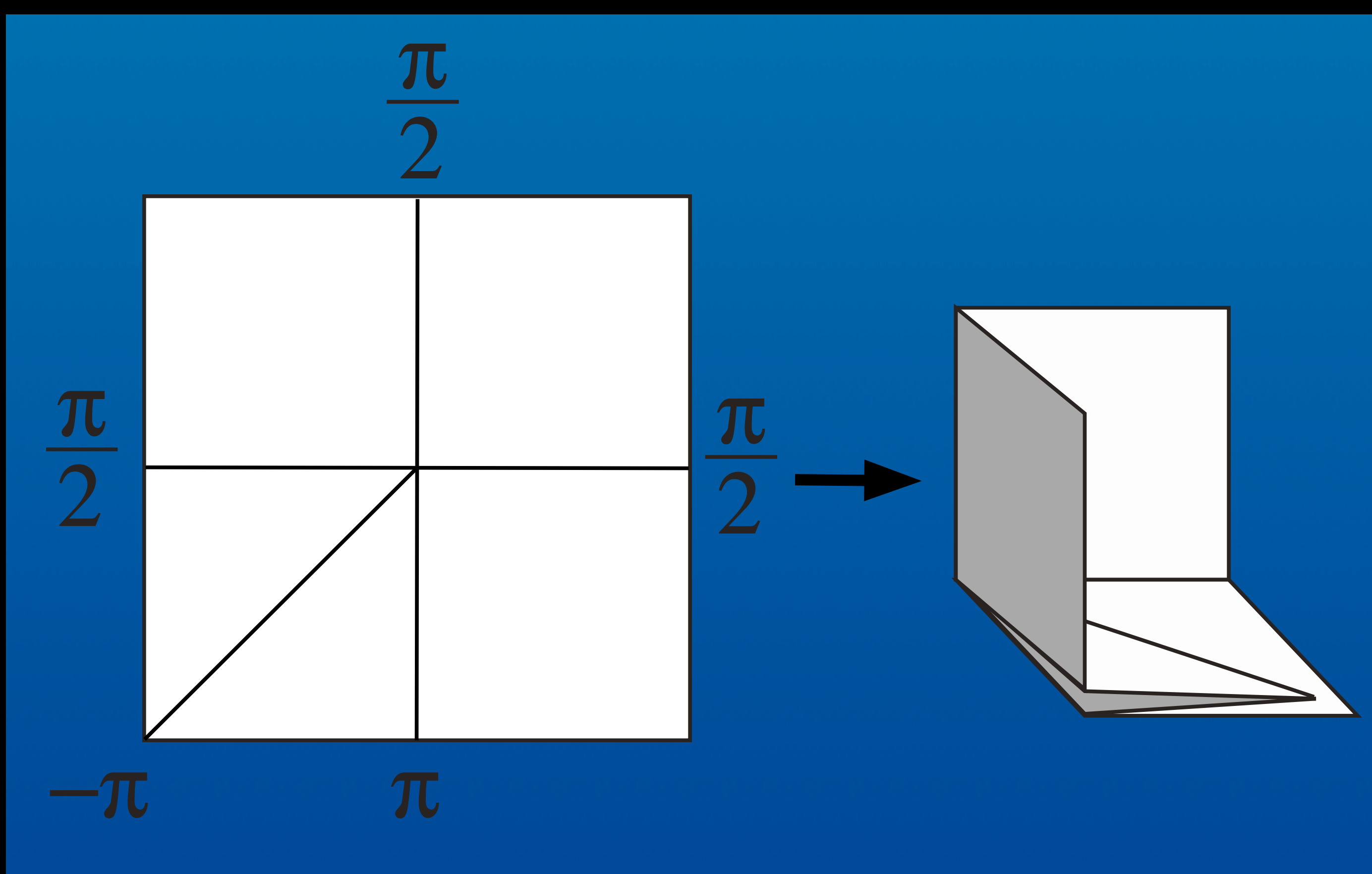
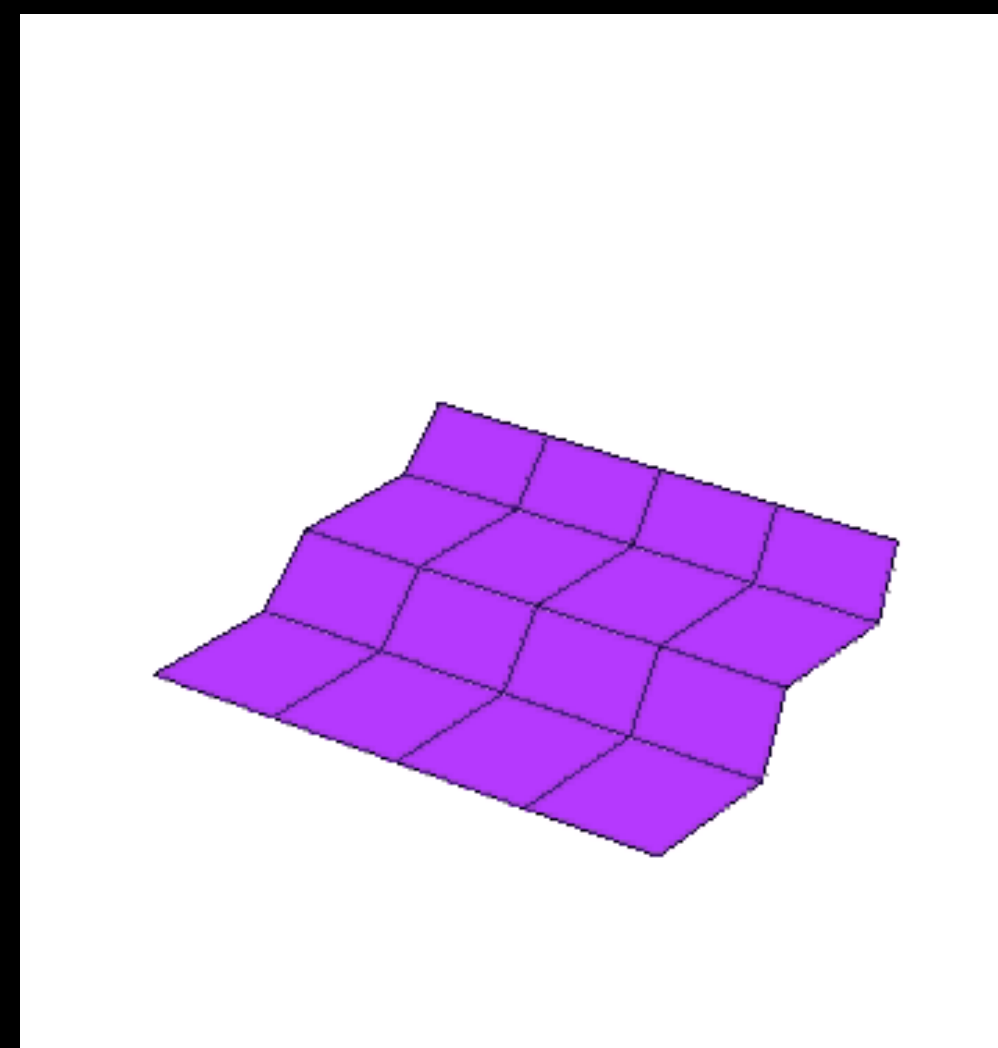
- origami geometric constructions
- combinatorial geometry of origami
- matrix models and rigid foldability





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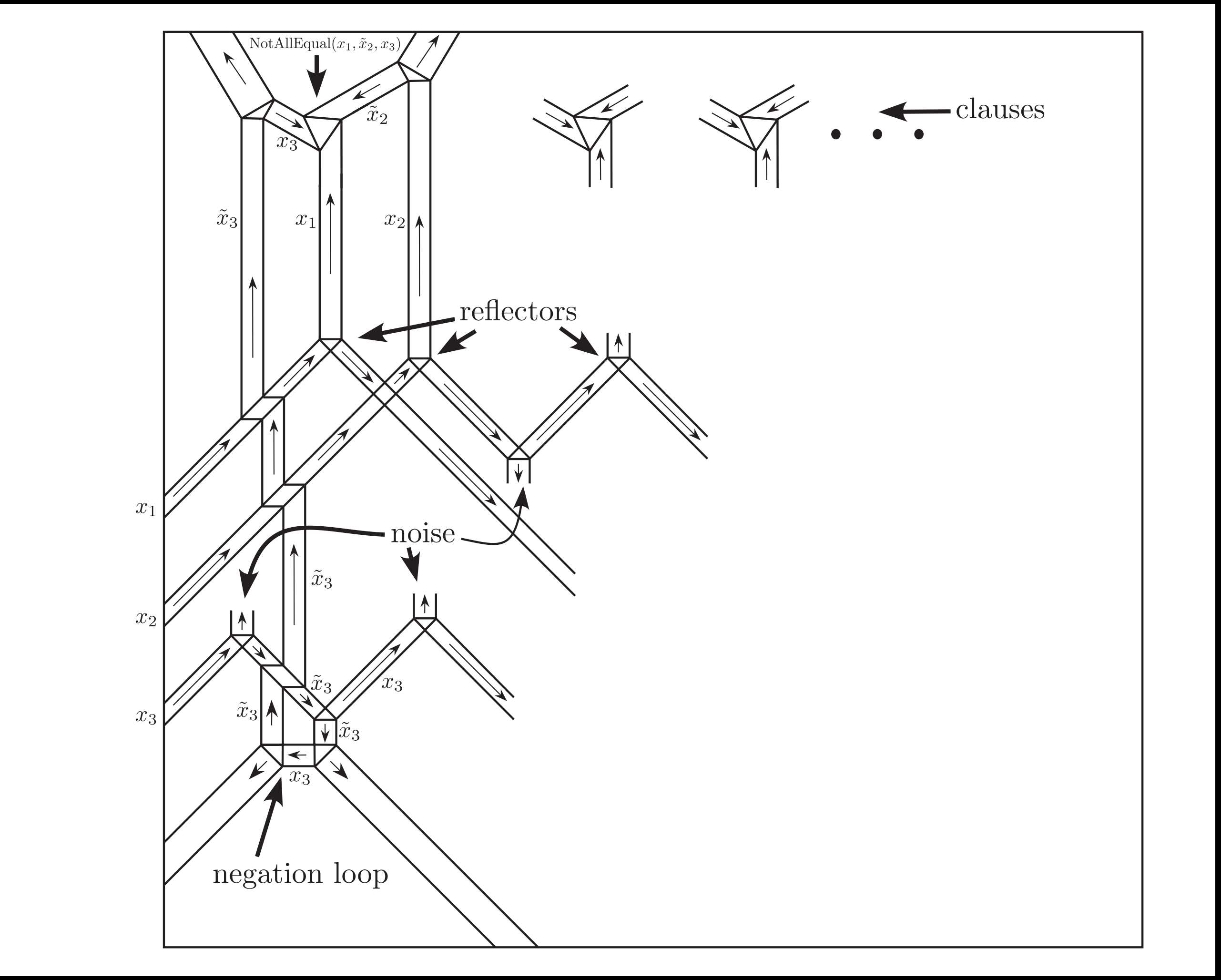
- computational complexity of origami



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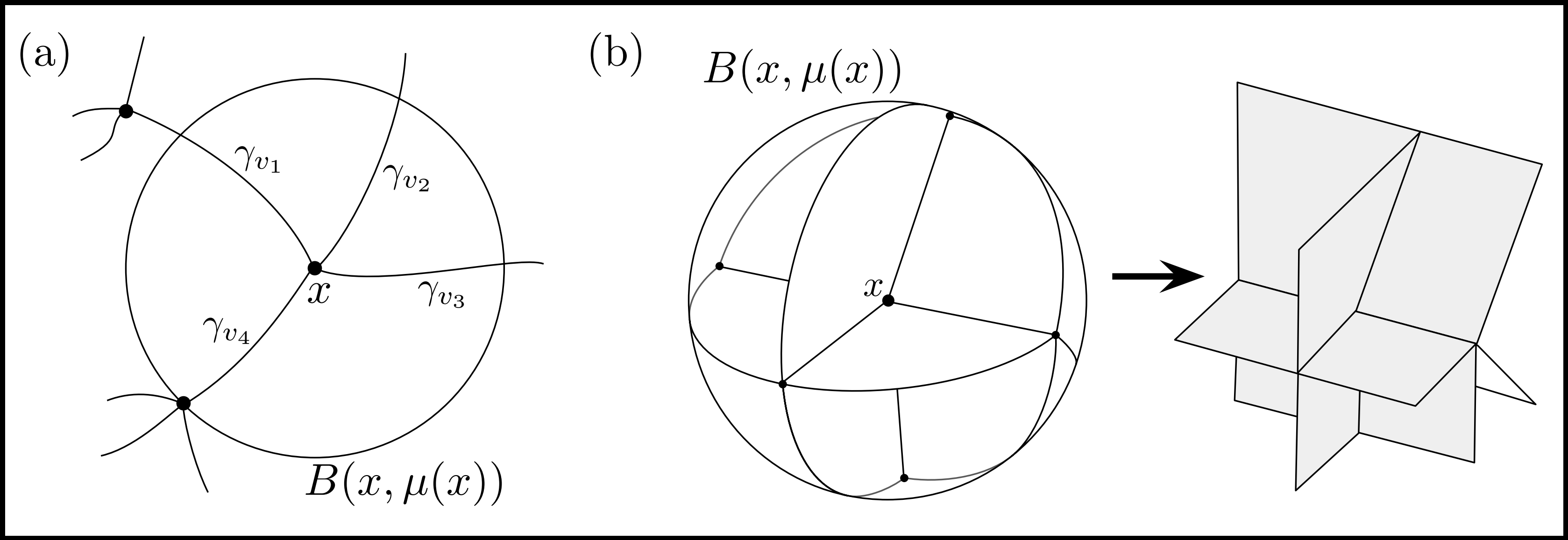
- folding manifolds



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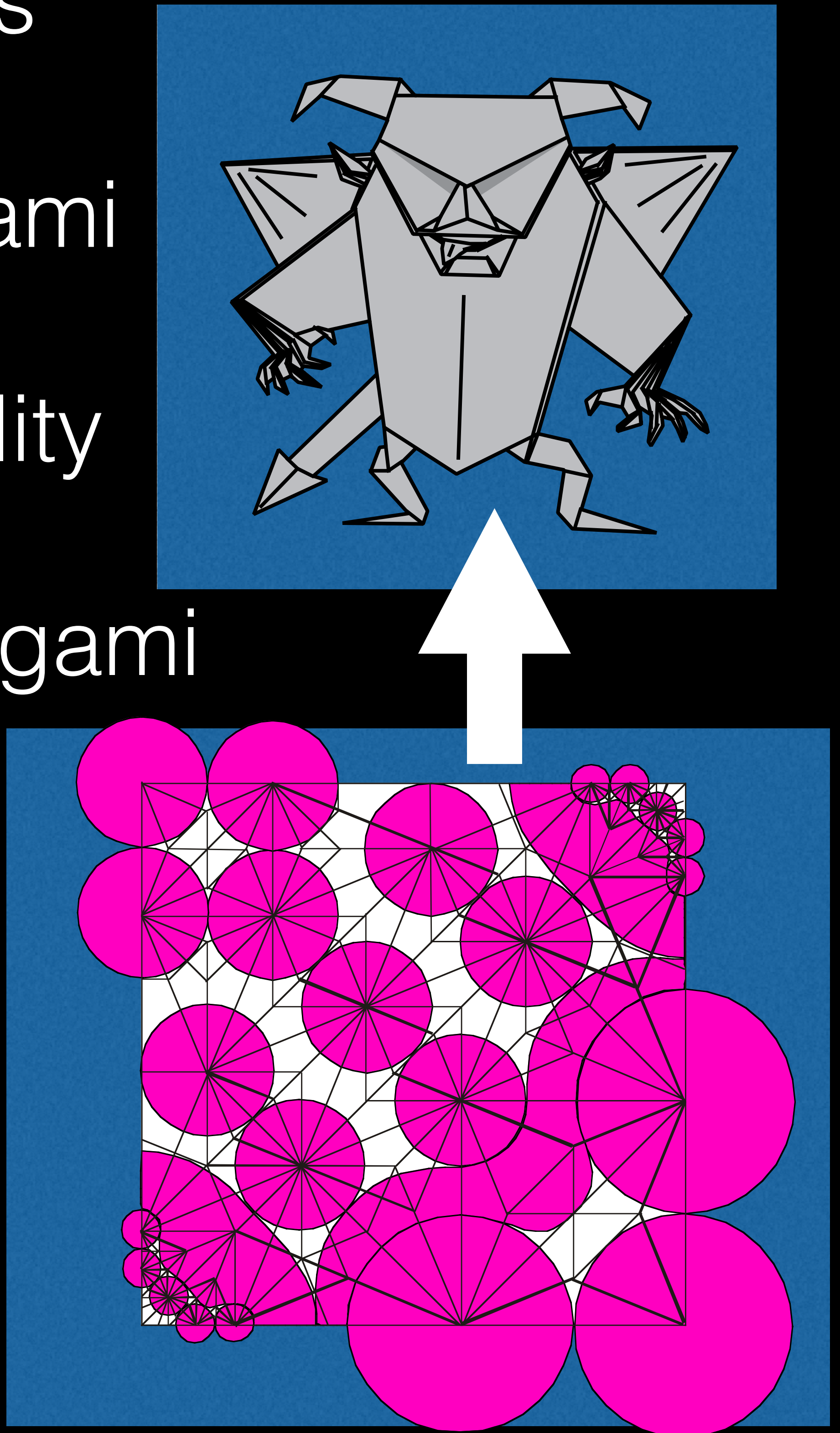

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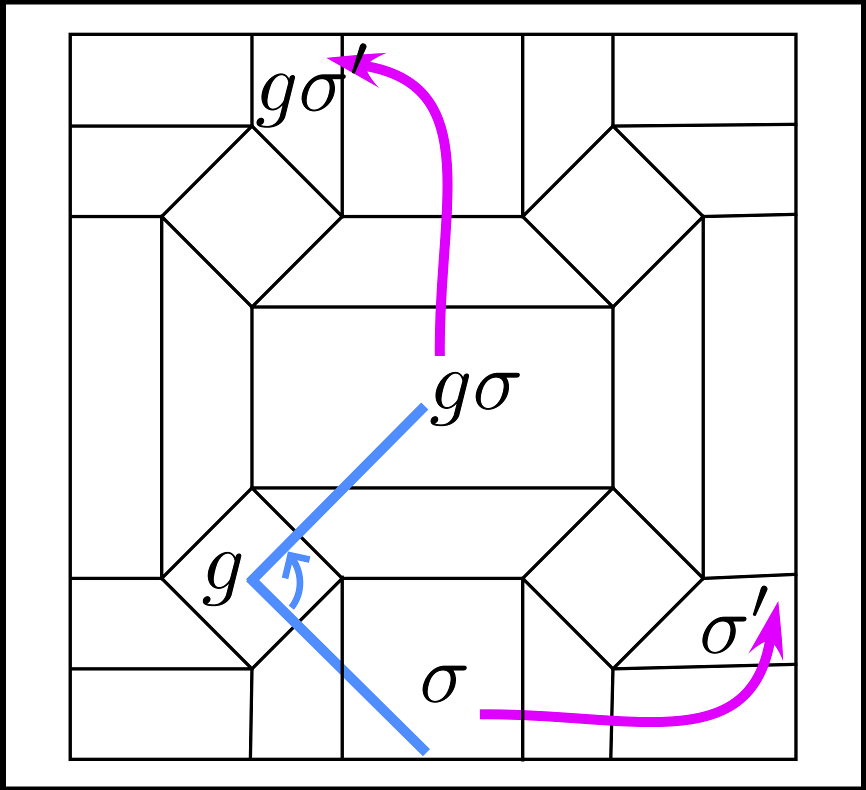
- origami algebra



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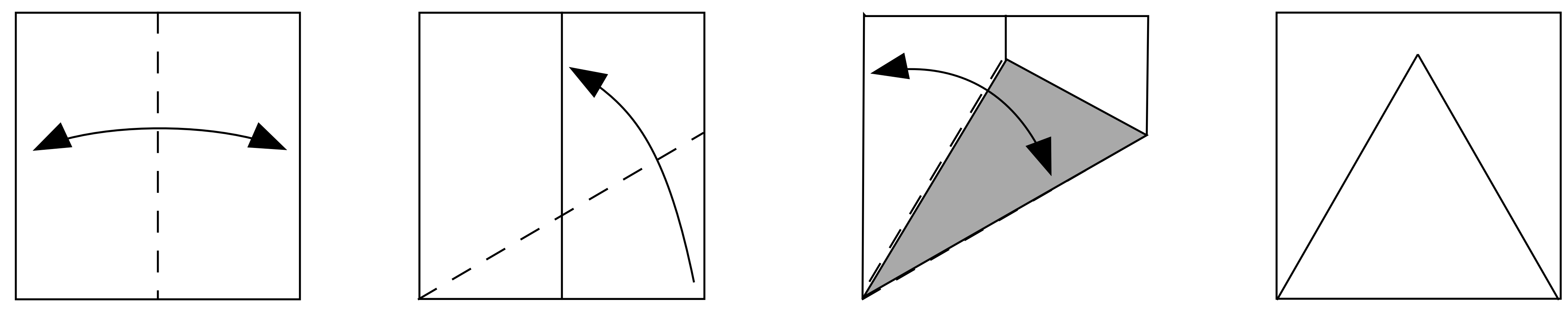
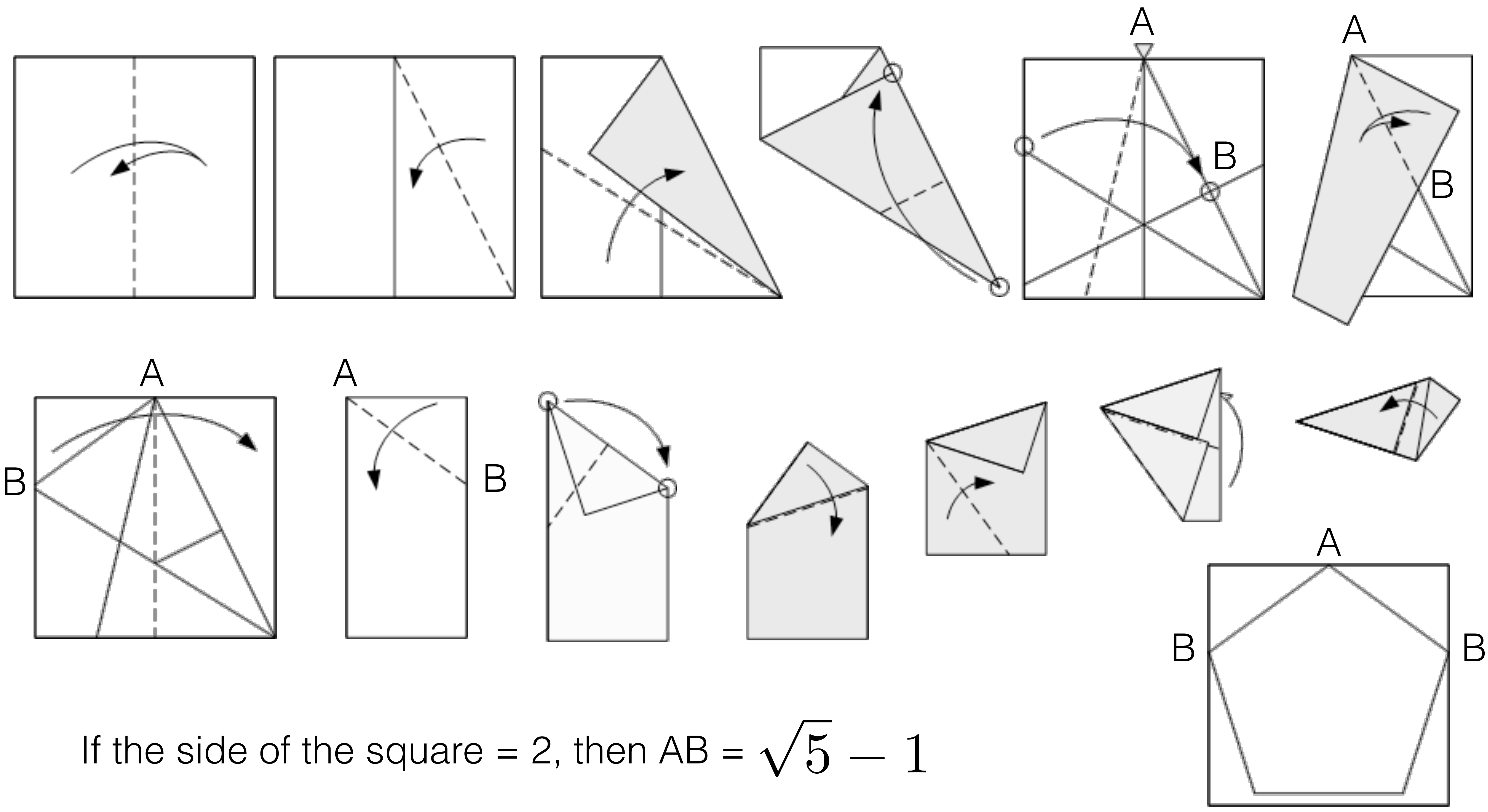
- origami algebra
- modular origami
- applications
- curved folds

Folding a Equilateral Triangle



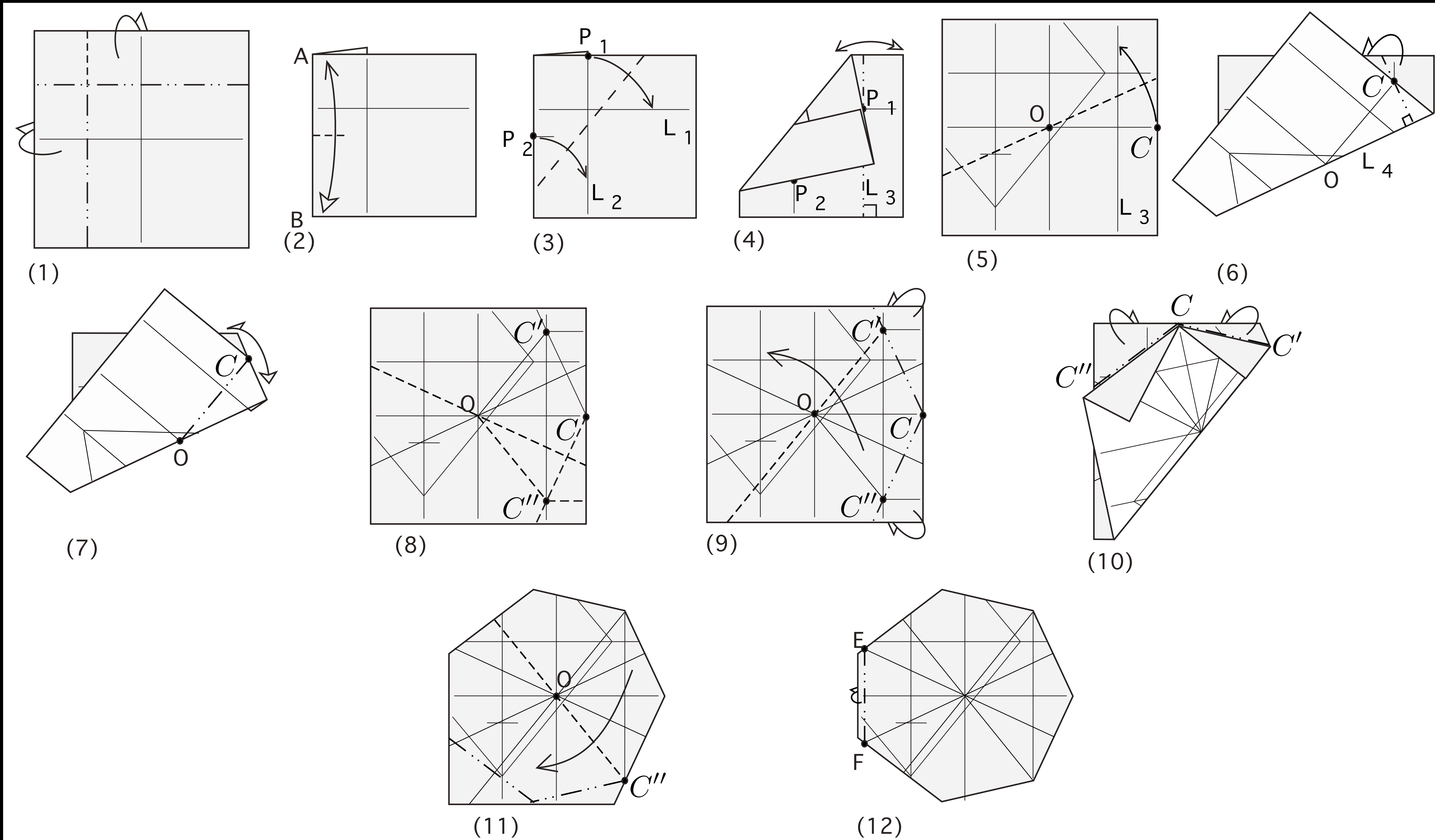
Folding a Regular Pentagon



If the side of the square = 2, then AB = $\sqrt{5} - 1$

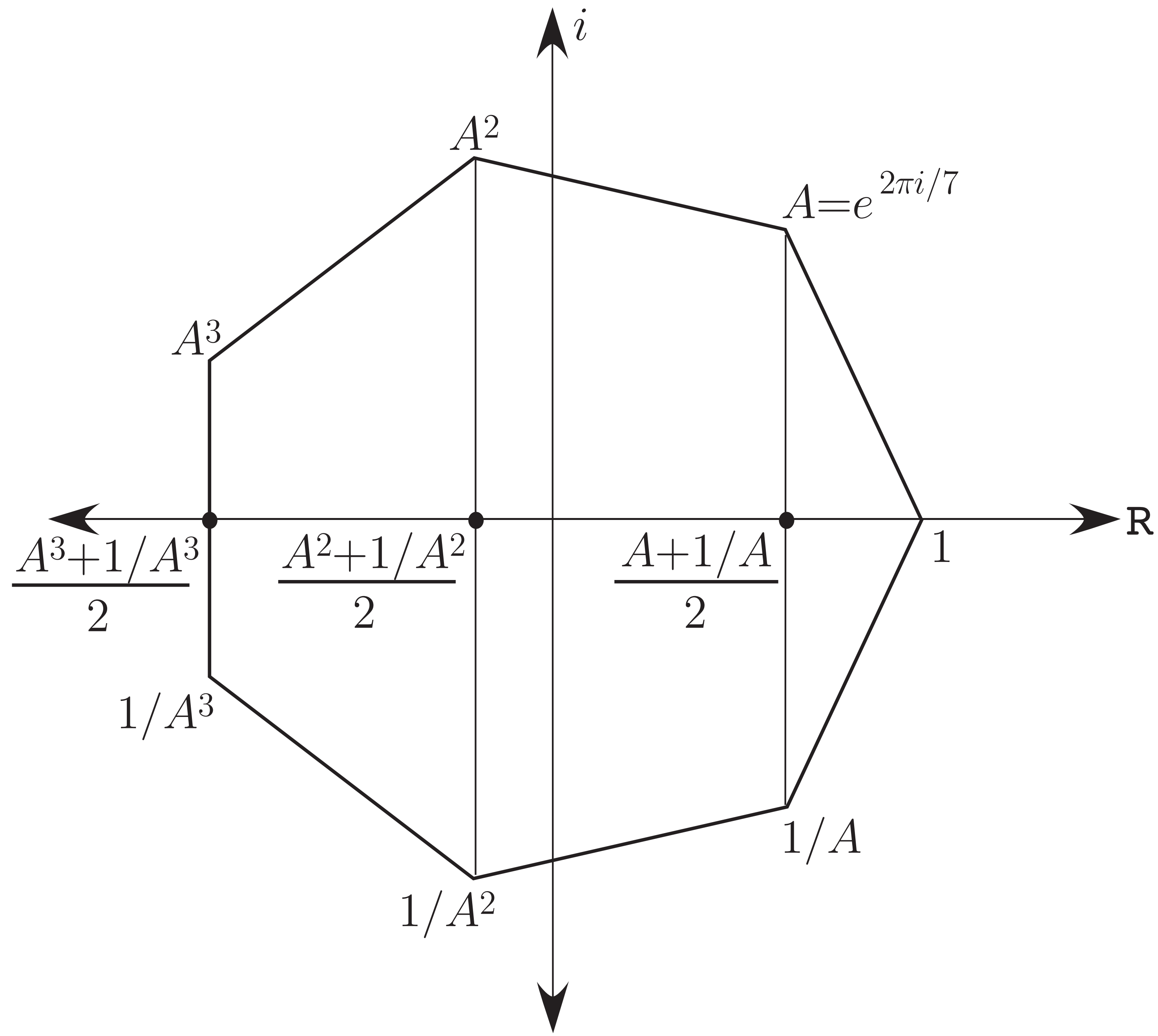
method by David Chandler

Folding a Regular Heptagon

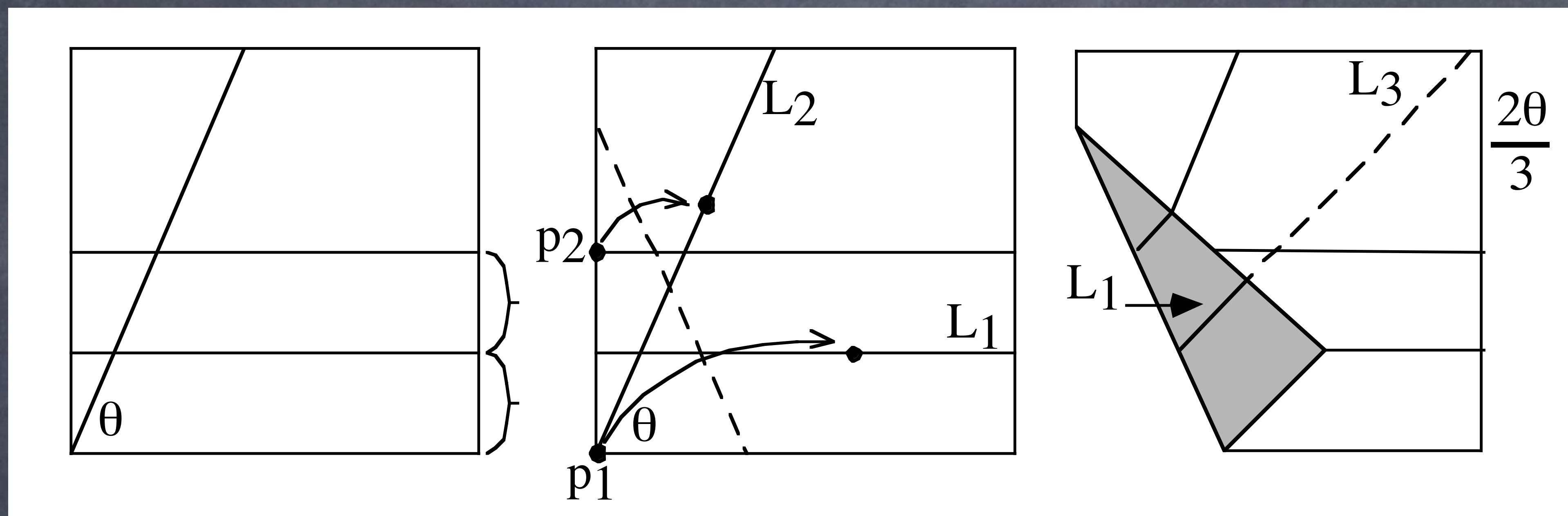


method by Hull

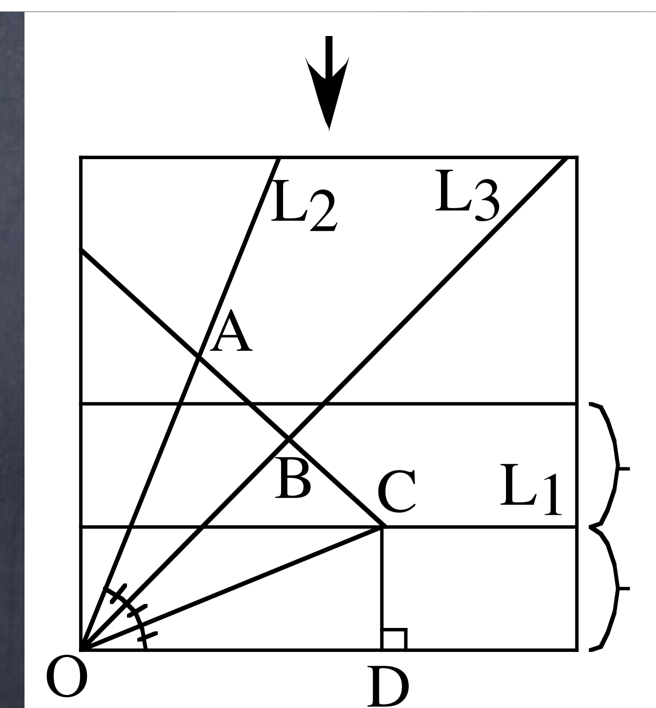
Folding a Regular Heptagon



Origami angle trisection



Proof:



credit: Hisashi Abe, 1980

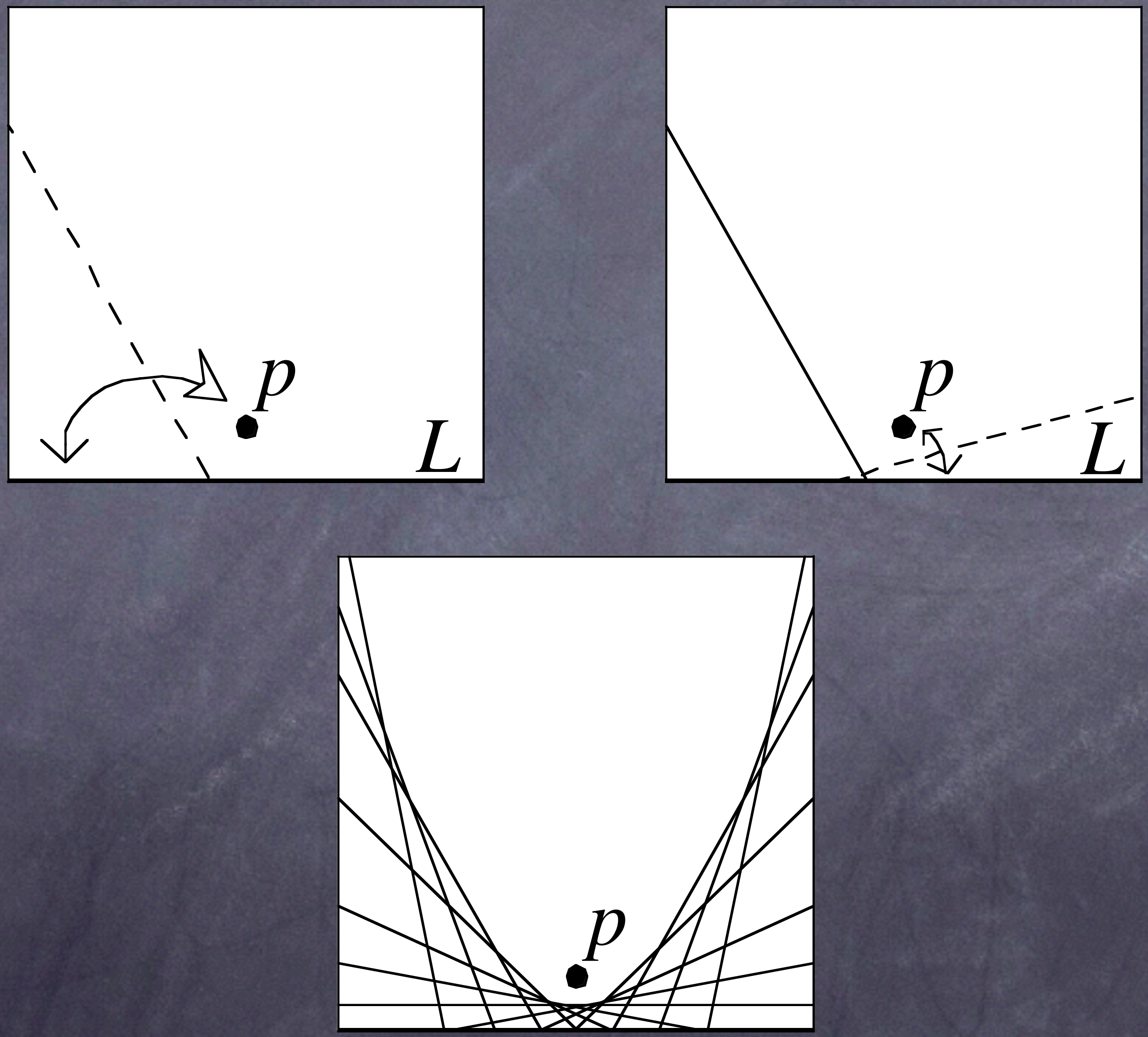
Straightedge and Compass basic operations

- Given two points P_1 and P_2, we can draw the line P_1P_2.
- Given a point P and a line segment of length r, we can draw a circle centered at P with radius r.
- We can locate intersection points, if they exist, between lines and circles.

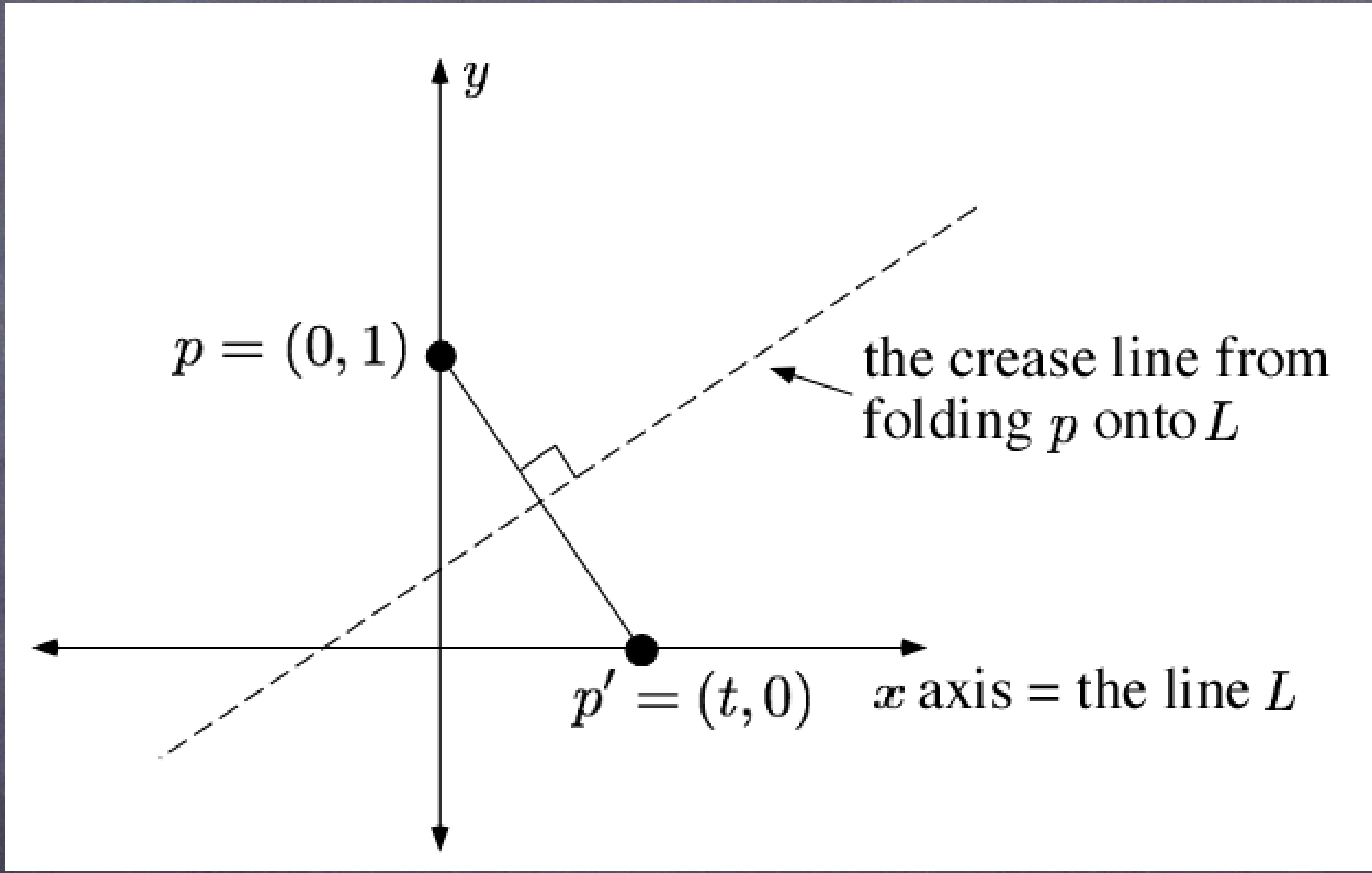
What are the Basic Operations of Origami?

- Given two points P_1 and P_2, we can fold the crease line P_1P_2.
- Given two points P_1 and P_2, we can make a crease that puts P_1 onto P_2.
- Given two lines L_1 and L_2, we can make a crease that puts L_1 onto L_2.
- and so on.

Folding a point to a line



Folding a point to a line



A different way...

Given a family of curves $F(x, y, t) = 0$ in $\mathbb{R}^2$, its **envelope** consists of all points (x, y) such that that for some $t \in \mathbb{R}$,

$$F(x, y, t) = 0 \quad \text{and} \quad \frac{\partial}{\partial t} F(x, y, t) = 0$$

The envelope is a single curve that is tangent to all the curves in the family.

In our case we have $F(x, y, t) = t^2 - 2xt + 2y - 1$

So $\frac{\partial}{\partial t} F(x, y, t) = 0$ becomes $2t - 2x = 0$, or $x = t$.

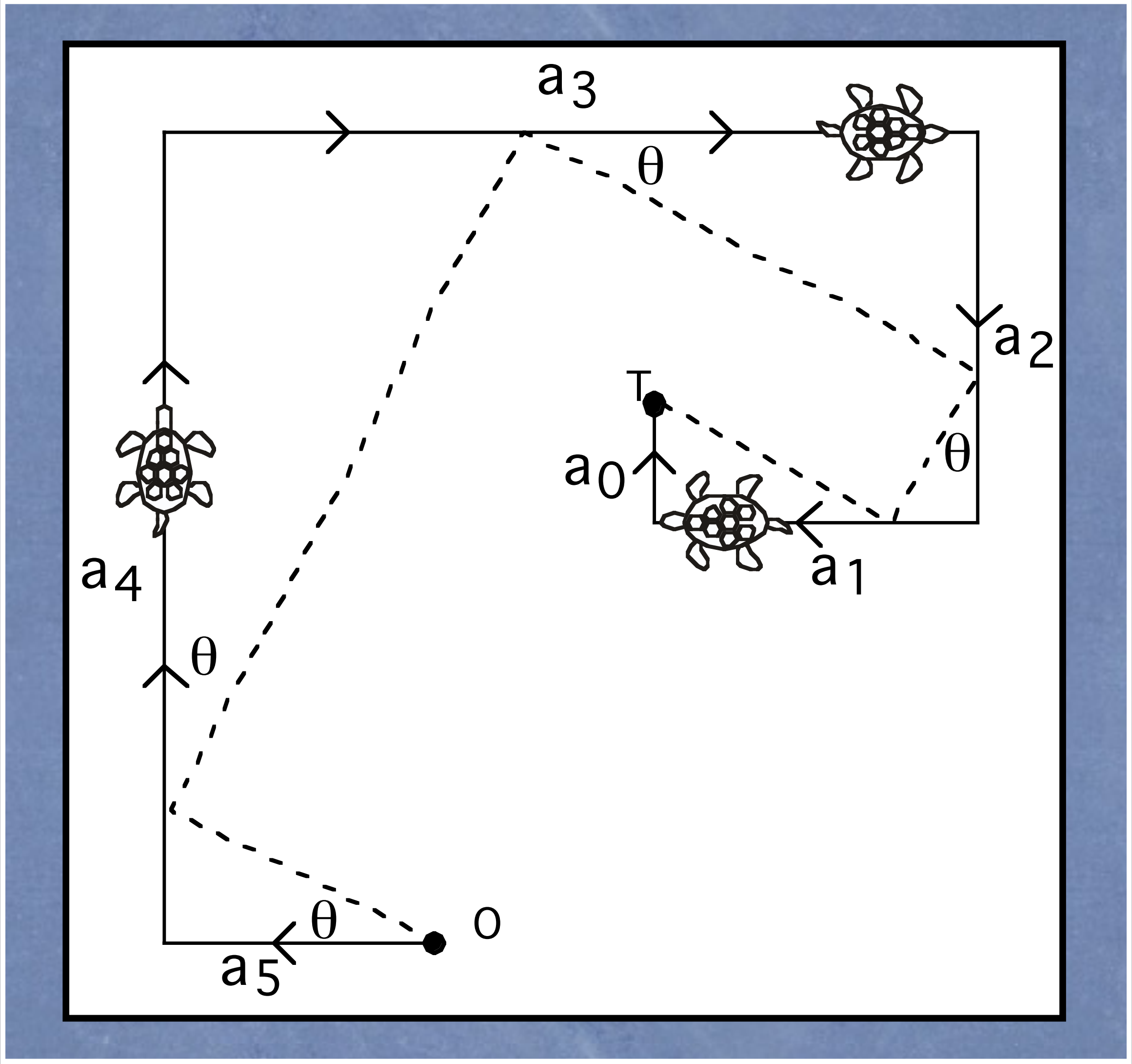
Combining this with $t^2 - 2xt + 2y - 1 = 0$ gives

$$-x^2 + 2y - 1 = 0 \quad \text{or} \quad y = \frac{1}{2}x^2 + \frac{1}{2}$$

Lill's Method

Lill's geometric method for finding real roots of any polynomial:

$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$



Start at O, go a_n, turn 90°, go a_{n-1}, turn 90°, etc, ending at T.

Then shoot from O with an angle θ, bouncing off the walls at right angles, to hit T.

Then $x = -\tan\theta$ is a root.

(Lill, 1867)

Lill's Method

Why does Lill's method work?

$P_nQ_{n-1} / a_n = \tan\theta = -x$

So $P_nQ_{n-1} = -a_nx$

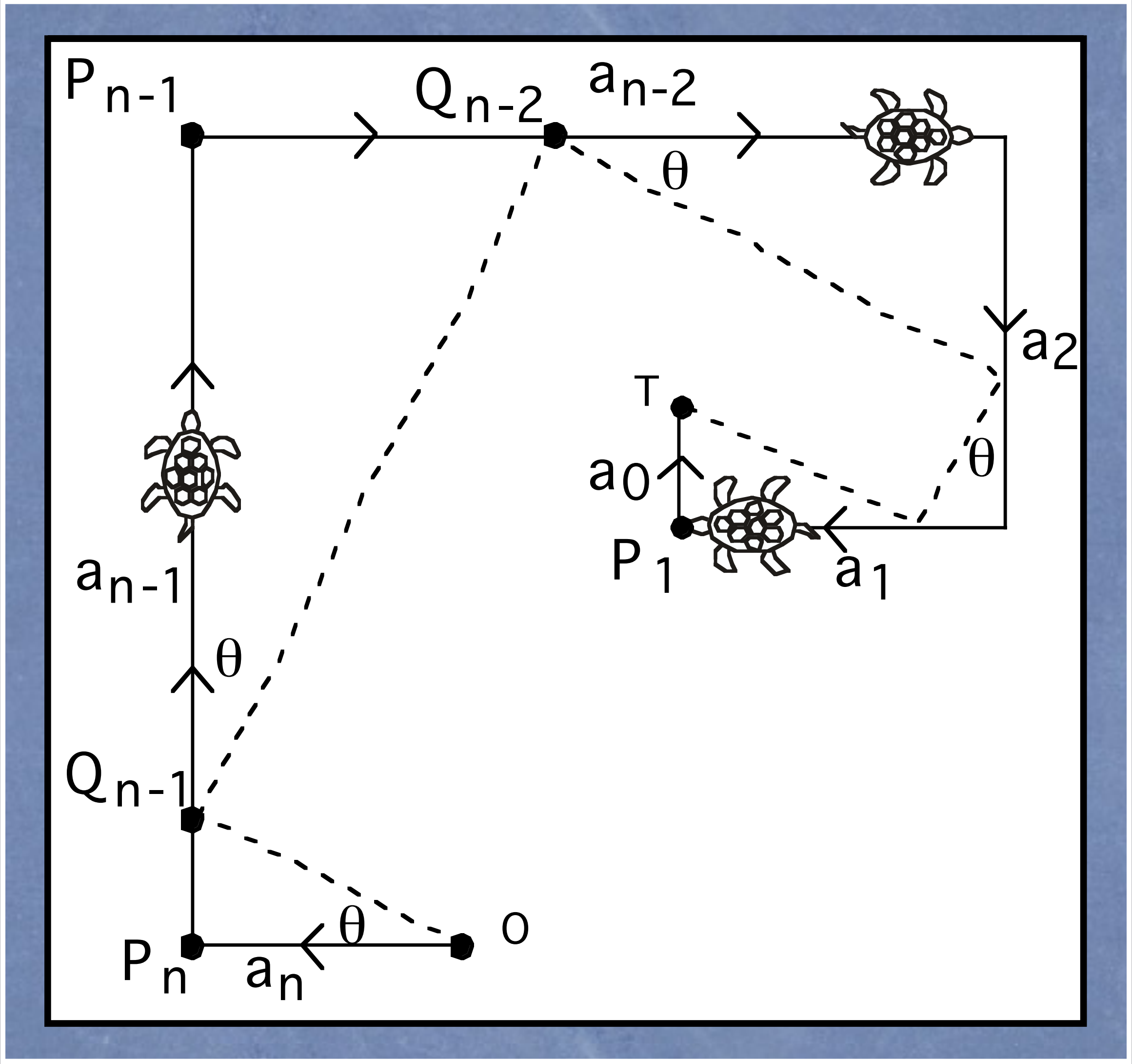
$P_{n-1}Q_{n-2} / (a_{n-1} - P_nQ_{n-1}) = -x$

So $P_{n-1}Q_{n-2} = -x(a_{n-1} + a_nx)$

Similarly,

$P_{n-2}Q_{n-3} = -x(a_{n-2}+x(a_{n-1}+a_nx))$

Continuing...



$a_0 = P_1T = - a_1x - a_2x^2 - \ldots - a_{n-1}x^{n-1} - a_nx^n$

or, $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0 = 0.$

(Lill, 1867)

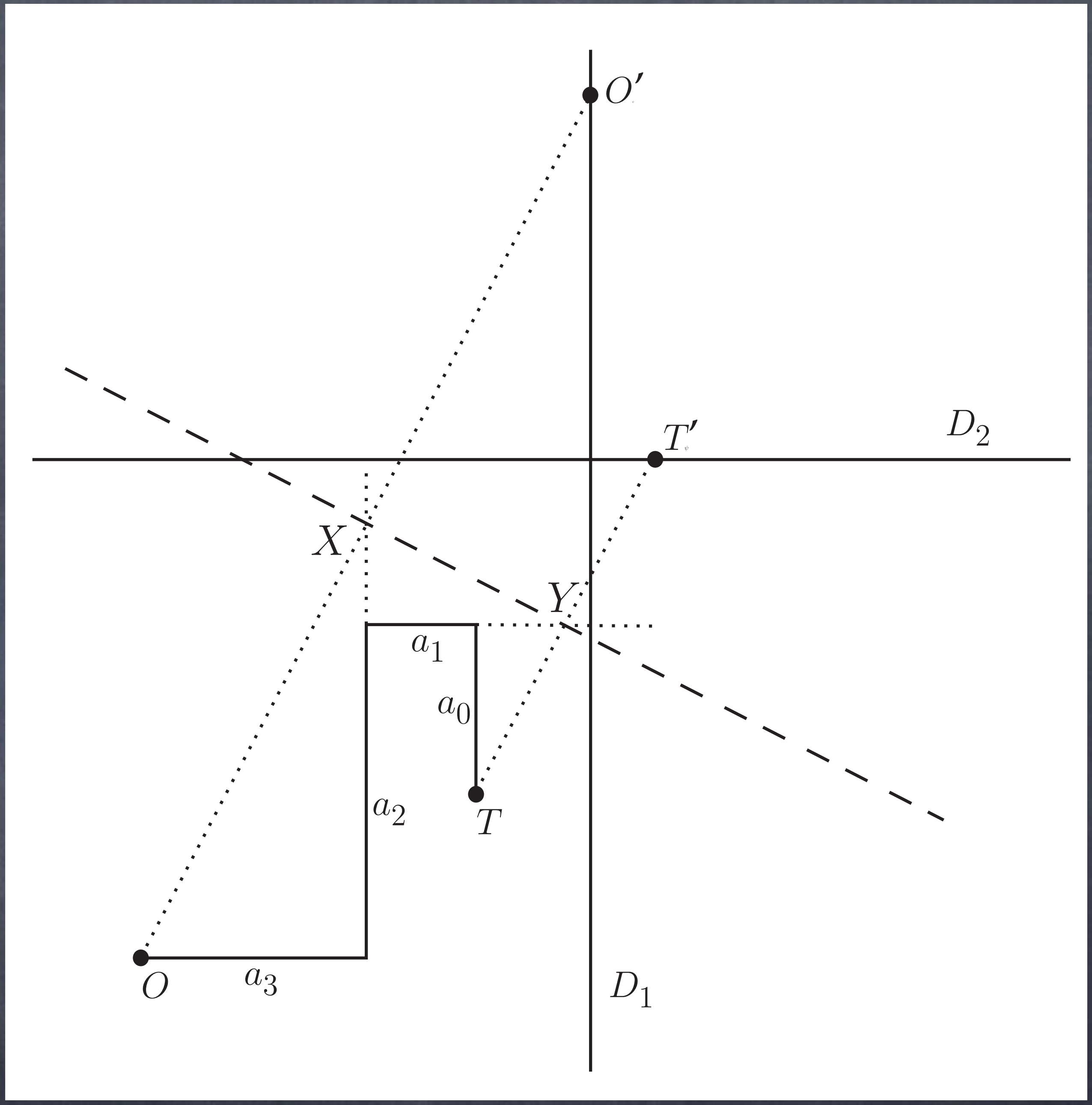
Beloch's use of Lill to Solve Cubics

Draw the turtle path a_3, a_2, a_1, a_0.

Draw D_1 at dist a_3 from and parallel to a_2.

Draw D_2 at dist a_0 from and parallel to a_1.

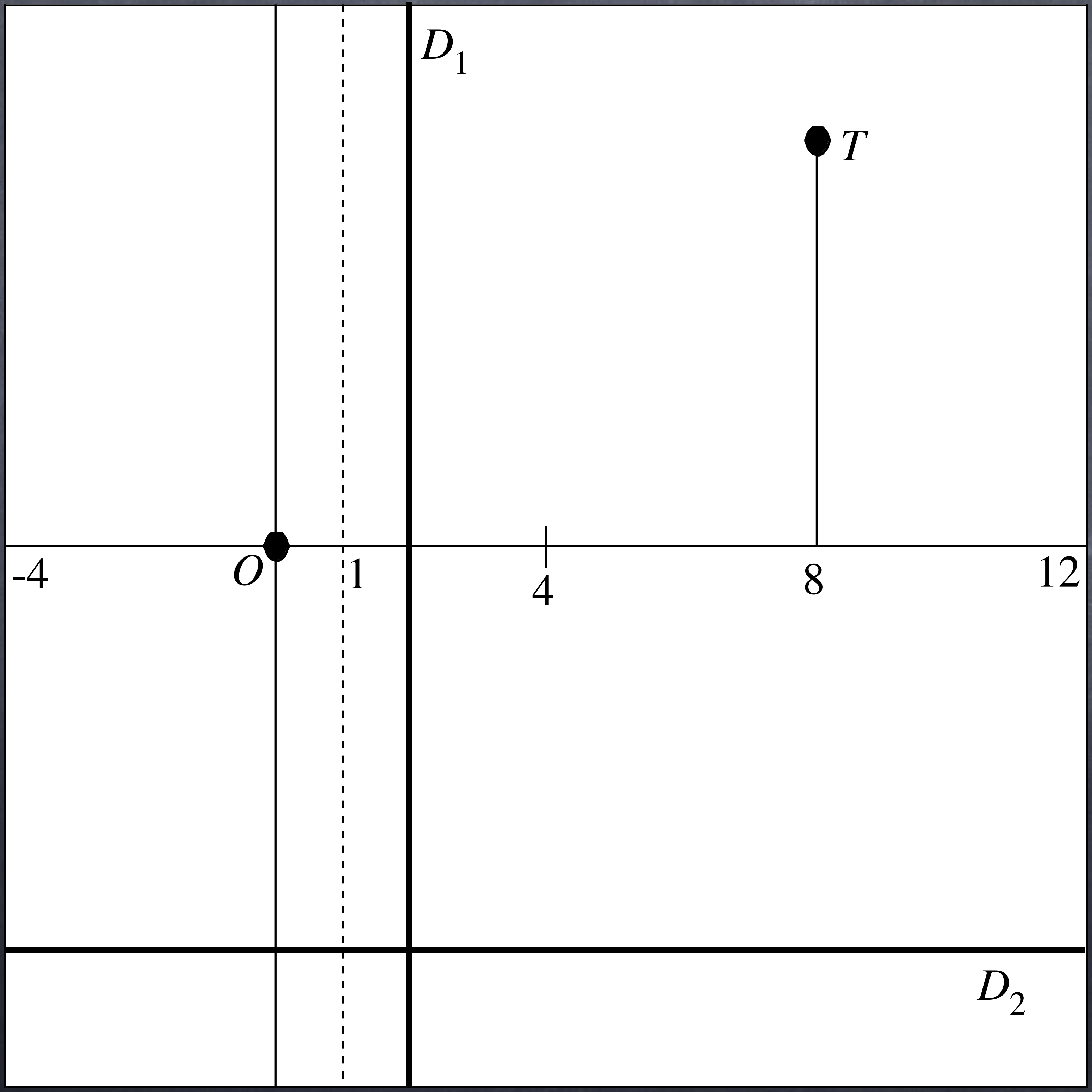
Then fold O onto D_1 and T onto D_2 at the same time.



(Beloch, 1936)

Origami can solve any cubic equation

Let's find the roots of the cubic $z^3 - 7z - 6$.



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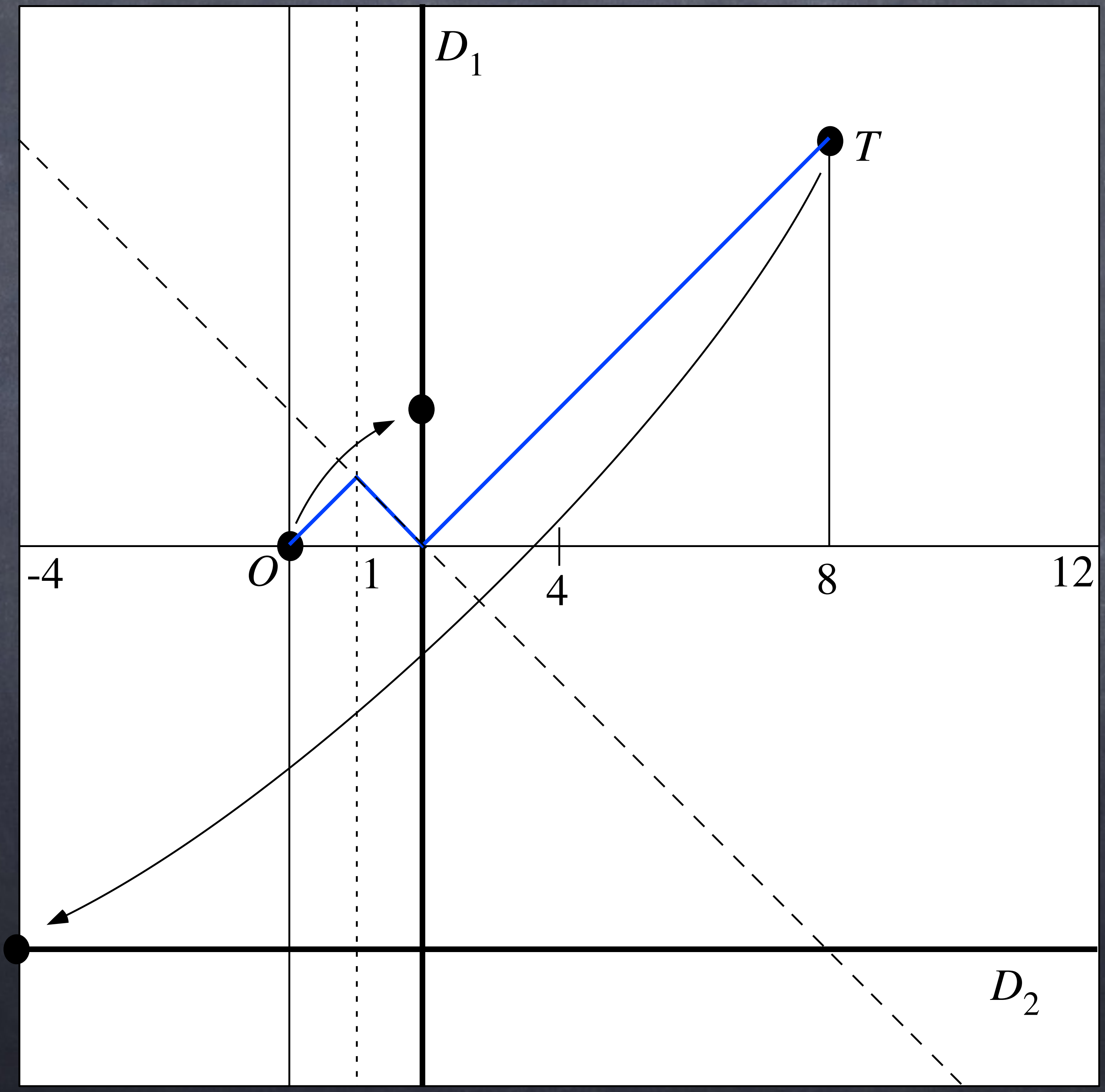

Here $\tan\theta = 1$, so $-\tan\theta = -1$ is a root.

Origami can solve any cubic equation

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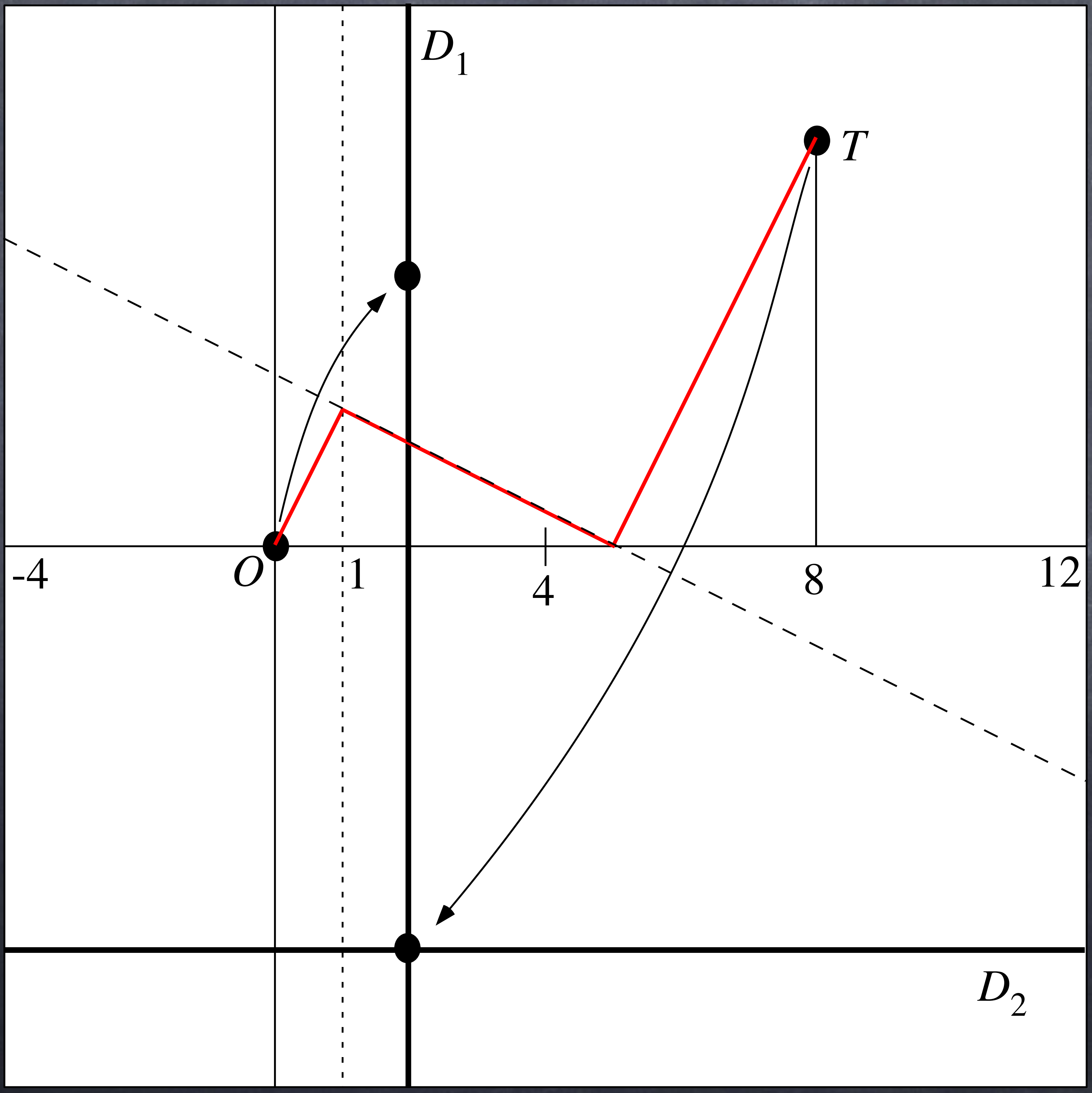
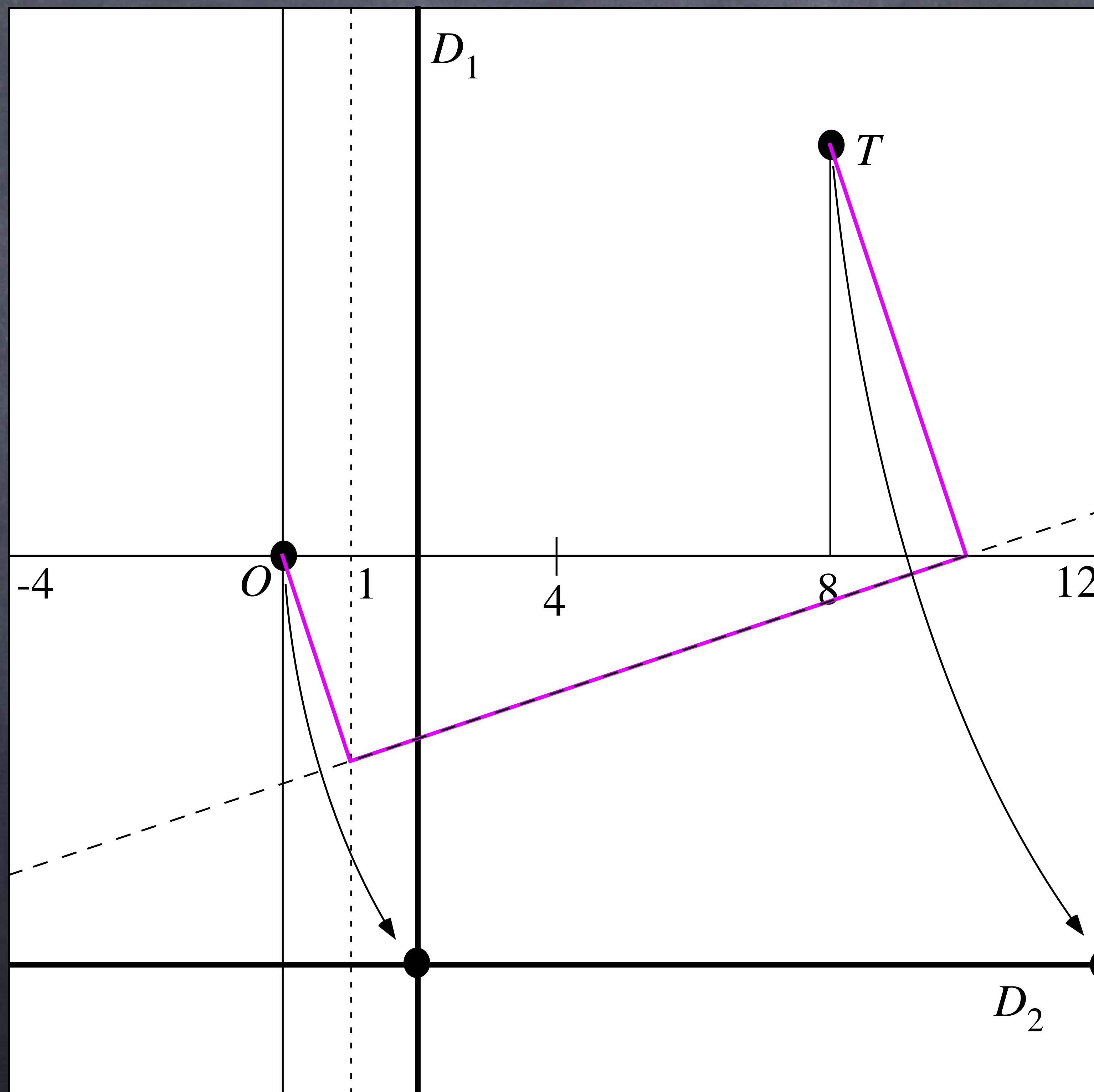


Here $\tan\theta = 2$, so $-\tan\theta = -2$ is a root.

Origami can solve any cubic equation

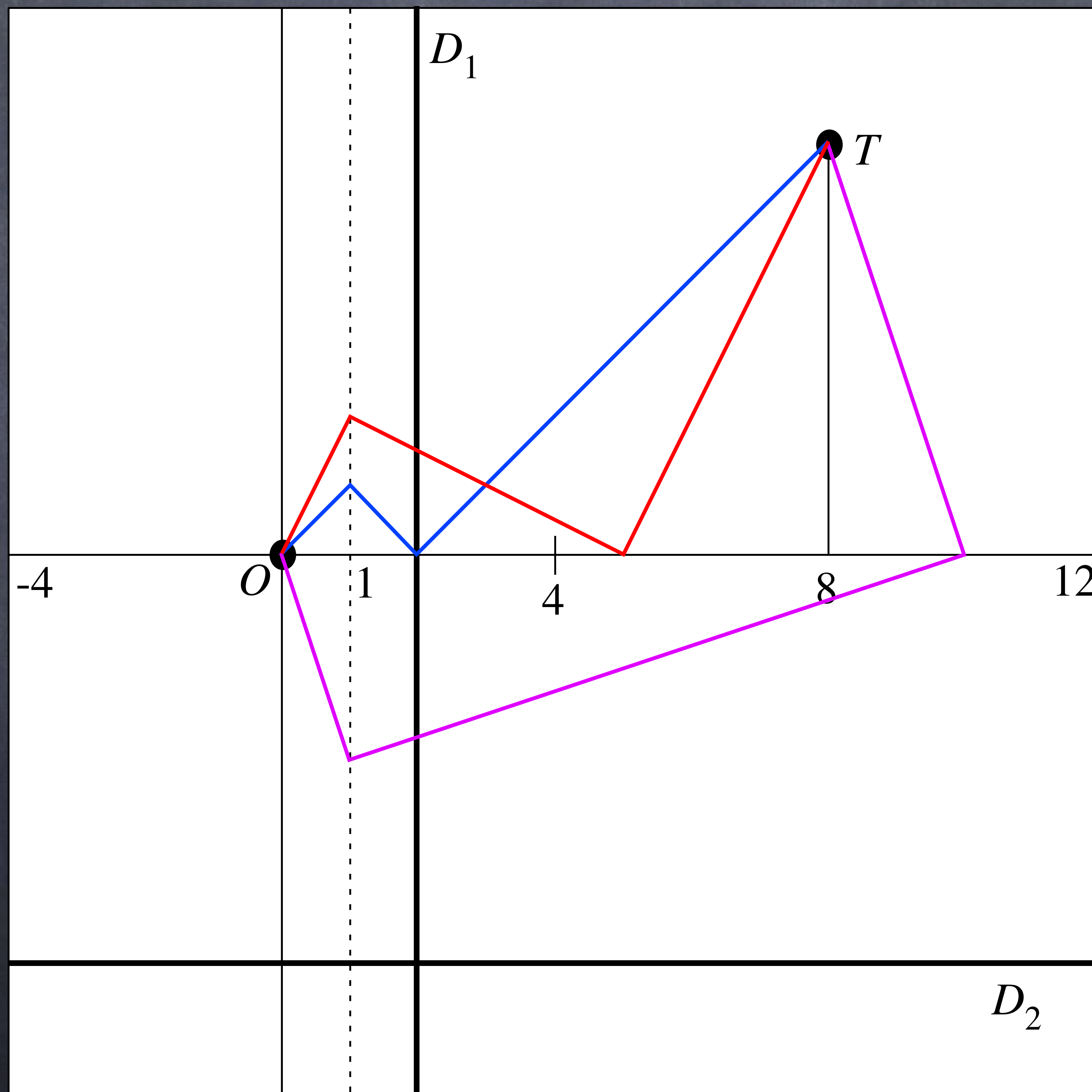
Let's find the roots of the cubic $z^3 - 7z - 6$.



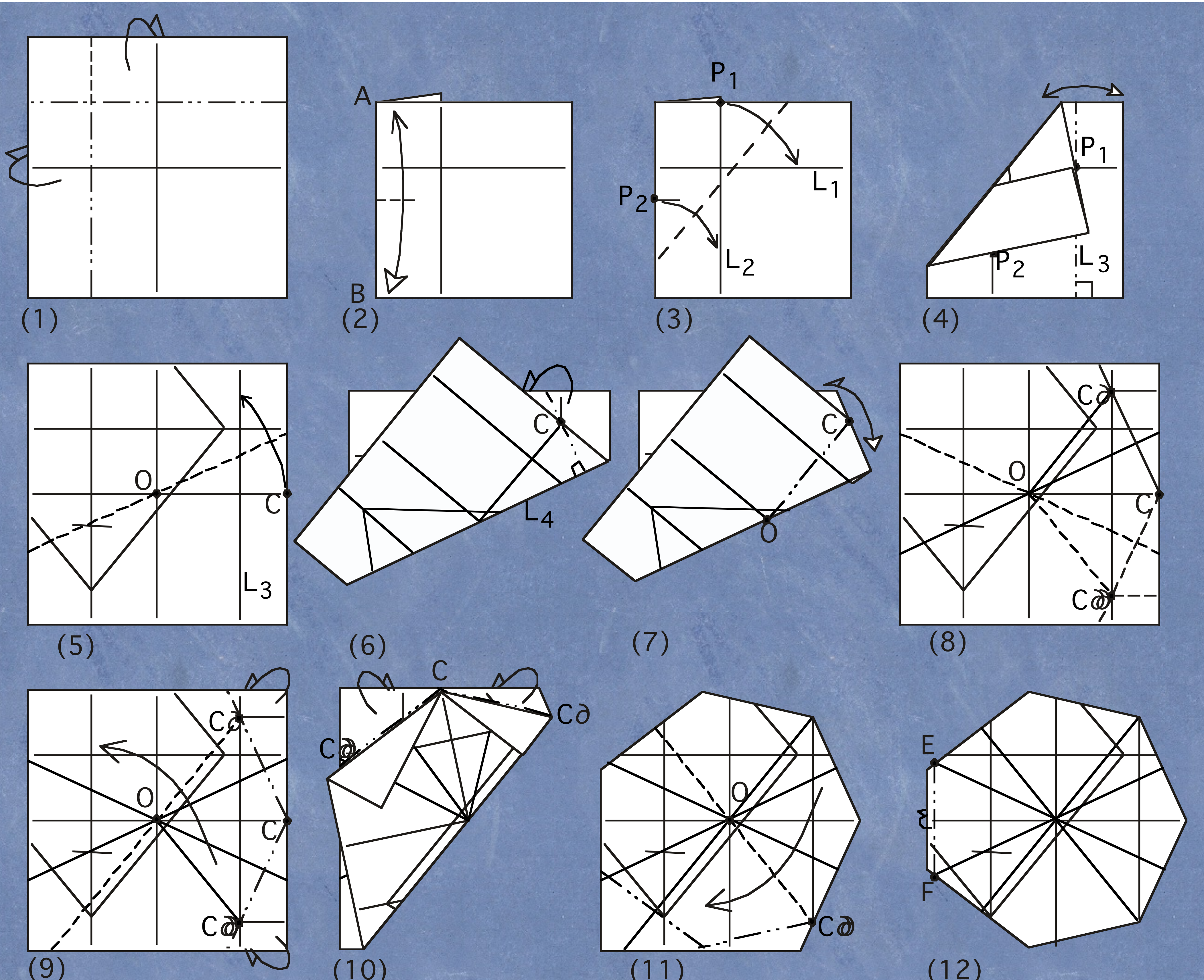
Here $\tan\theta = -3$, so $-\tan\theta = 3$ is a root.

Origami can solve any cubic equation

Let's find the roots of the cubic $z^3 - 7z - 6$.



How to fold a regular heptagon



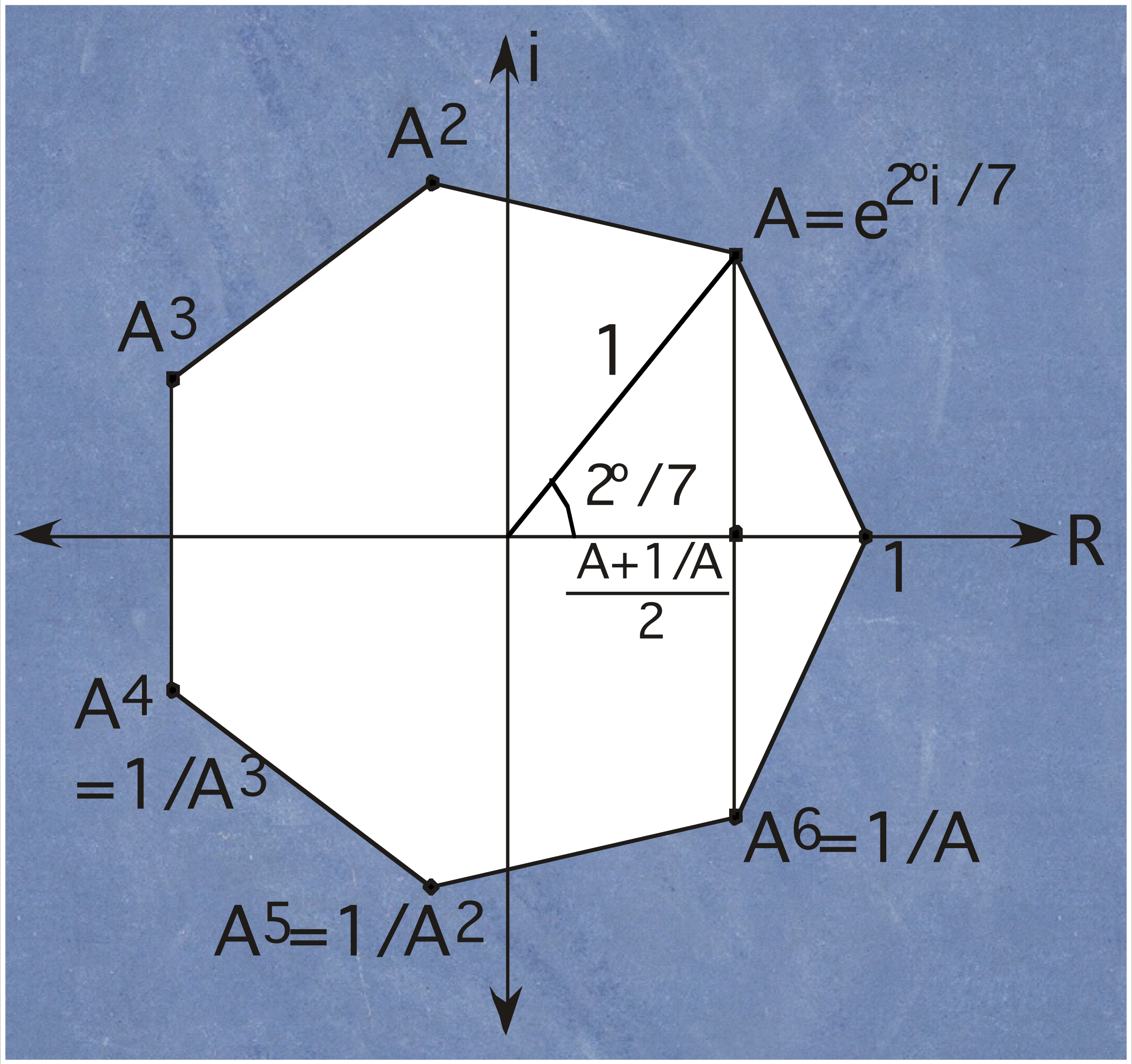
Heptagon -> cubic equation

The equation

$$x^3 + x^2 - 2x - 1 = 0$$

has as a solution

$$A + \frac{1}{A} = 2\cos(2\pi/7)$$

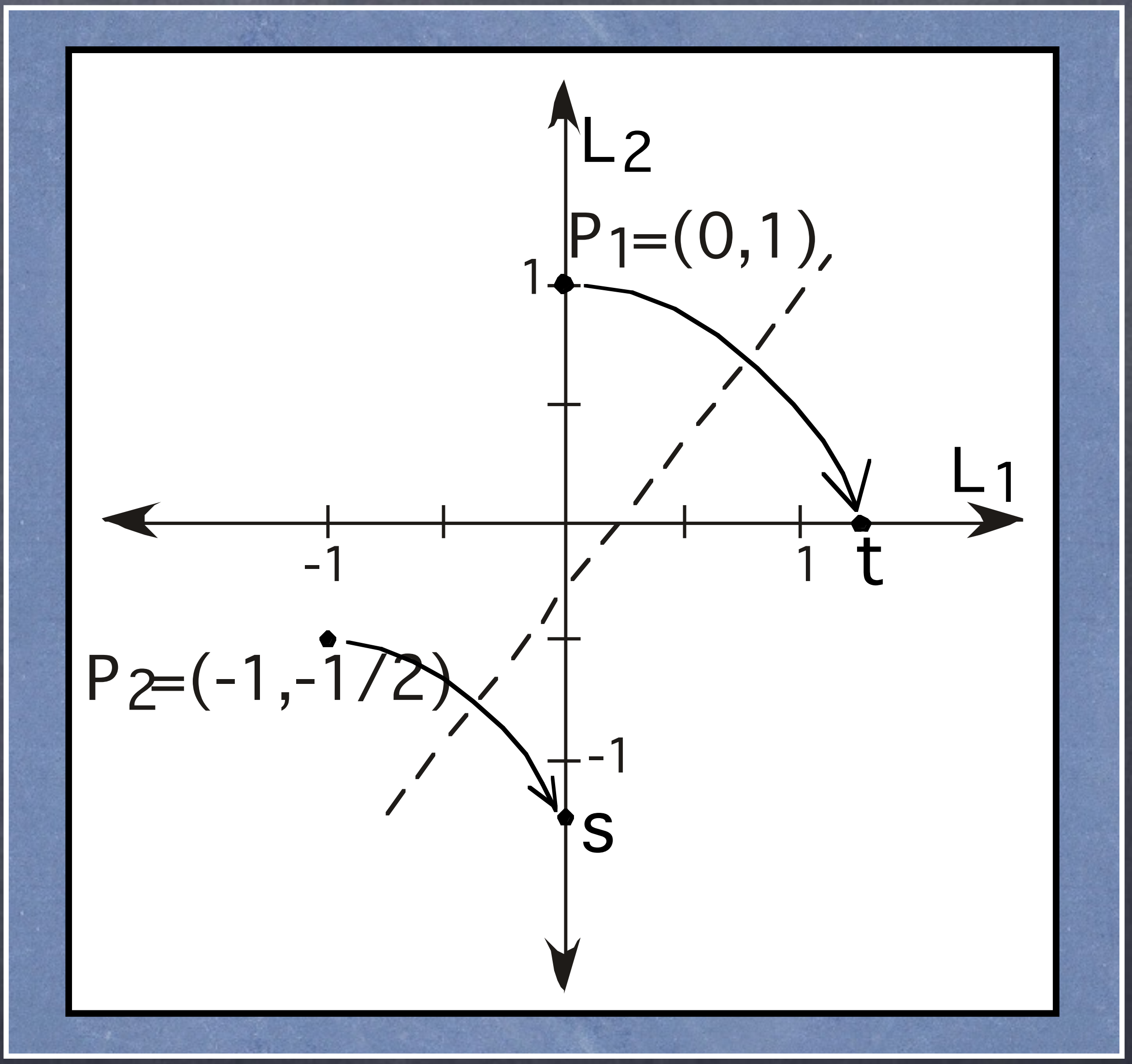


Therefore, solving this equation will give us $\frac{2\pi}{7}$

Task 2: Solve $x^3 + x^2 - 2x - 1 = 0$ by origami

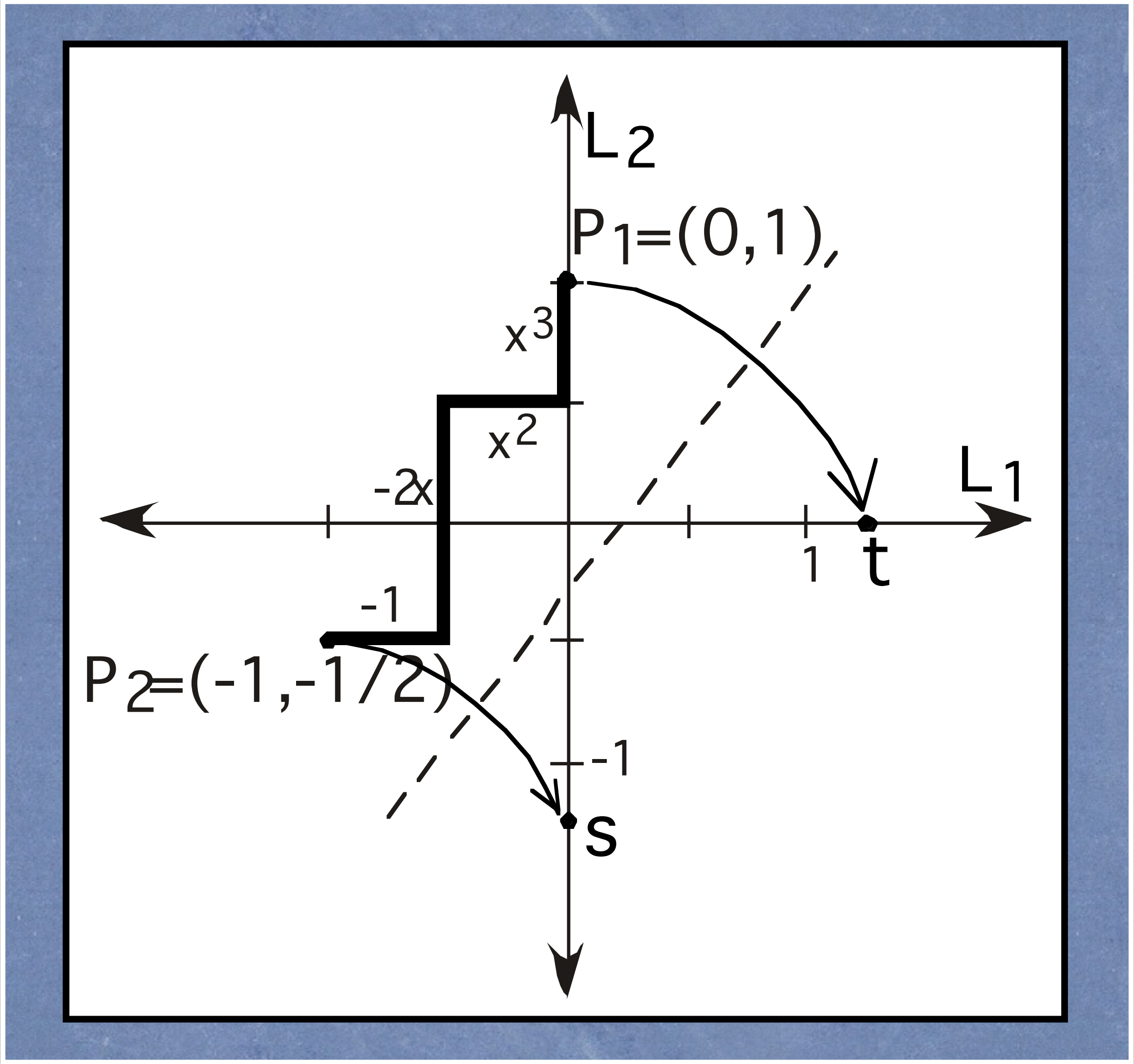
Fold $P_1 = (0,1)$ onto the x-axis and $P_2 = (-1,-1/2)$ onto the y-axis at the same time.

Claim: When we do this fold, P_1 gets folded to $t = 2\cos(2\pi/7)$ on the x-axis.



Task 2: Solve $x^3 + x^2 - 2x - 1 = 0$ by origami

Start with $P_1 = (0,1)$.
Make a "turtle diagram" from P_1 using the coefficients/2.
(Go, turn 90°, repeat.)



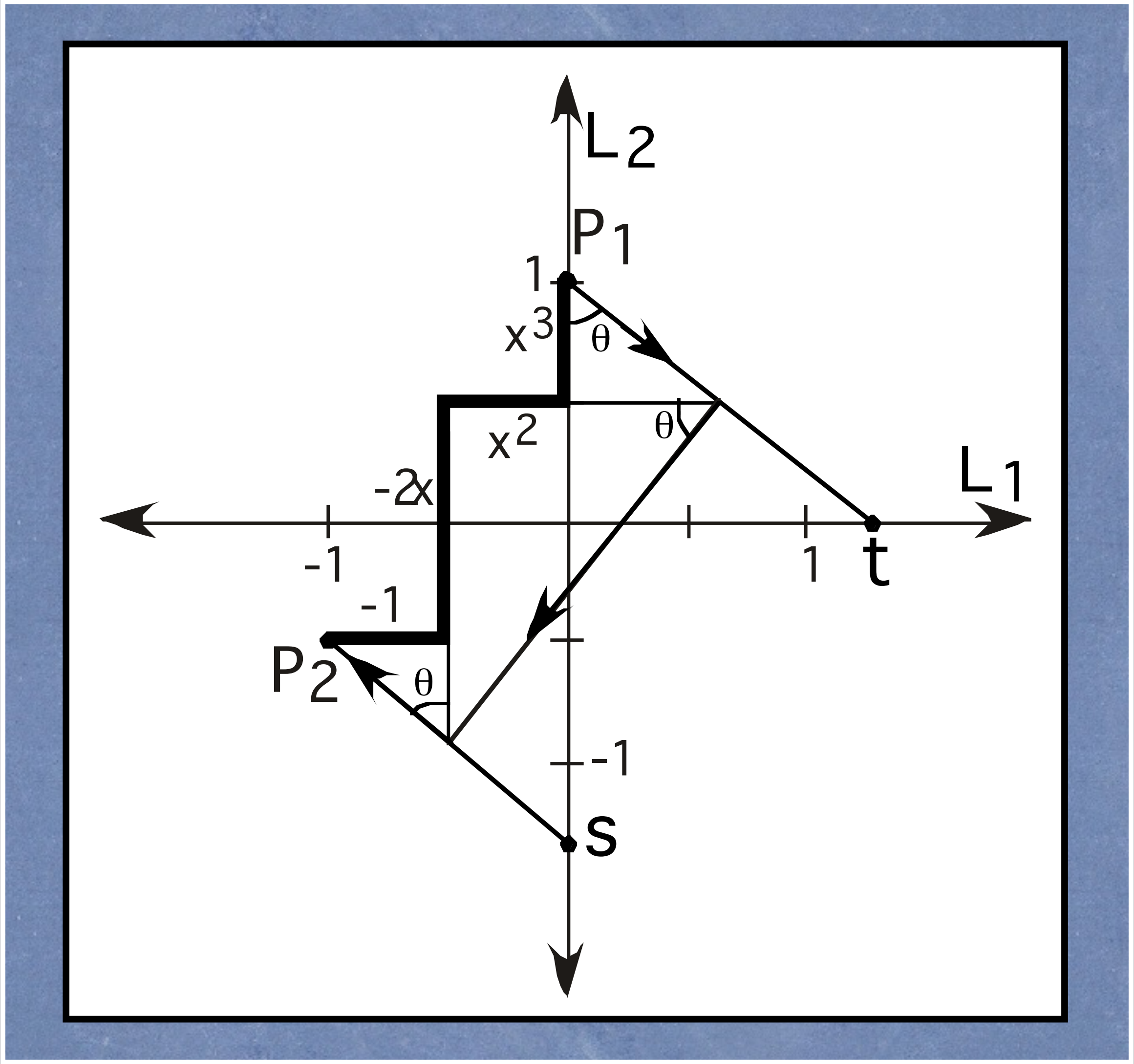
Task 2: Solve $x^3 + x^2 - 2x - 1 = 0$ by origami

How did we pick P_1 and P_2?

Then we "shoot" from P_1 by angle θ, bounce off the x^2 and $-2x$ lines, to hit P_2.

This gives us three similar triangles.

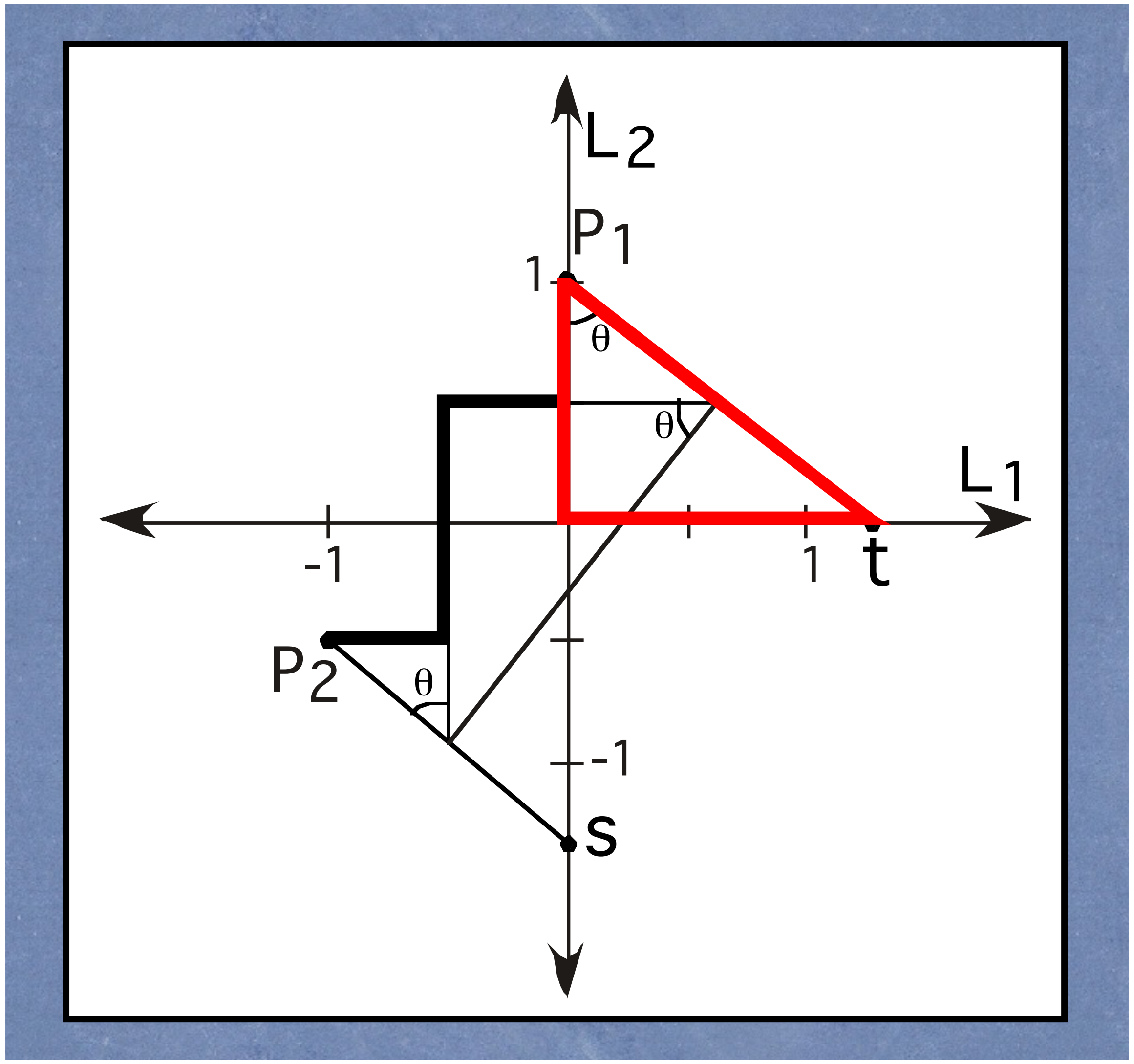
Also note that the hypothenuse of the biggest triangle is the crease line.



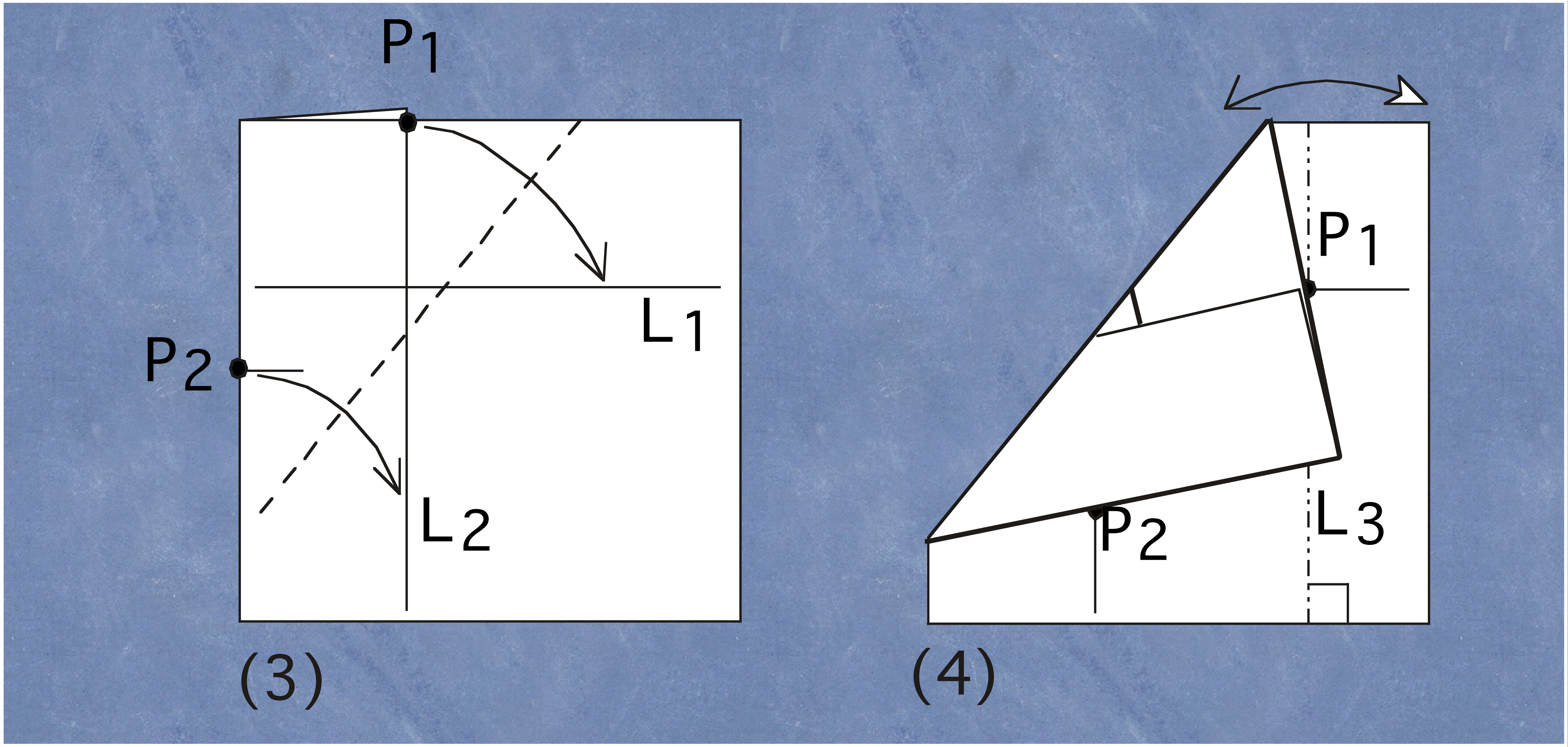
Task 2: Solve $x^3 + x^2 - 2x - 1 = 0$ by origami

Notice how the triangle made by P_1, t, and the origin verifies that

$t = \tan\theta = x = 2\cos(2\pi/7)$.



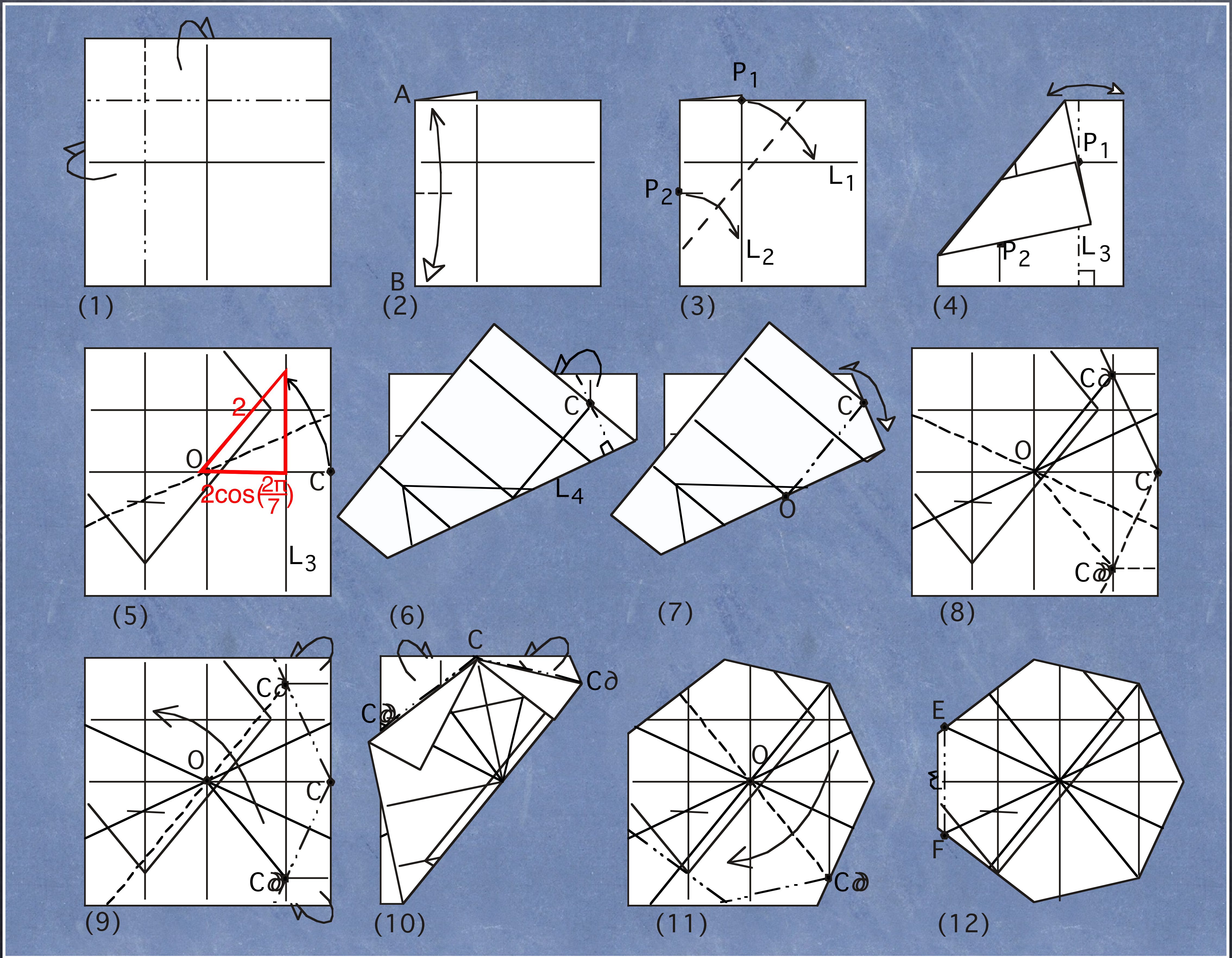
How we fold this



Step 3 has $P_1=(0,1)$ being folded to the x-axis and $P_2=(-1,-1/2)$ being folded to the y-axis.

In step 4 we mark P_1's location to get $t=2\cos(2\pi/7)$.

Does it make sense now?



The Algebraic Perspective

The set of constructible numbers under SE&C is the smallest subfield of $\mathbb{C}$ (complex #s) that is closed under square roots.

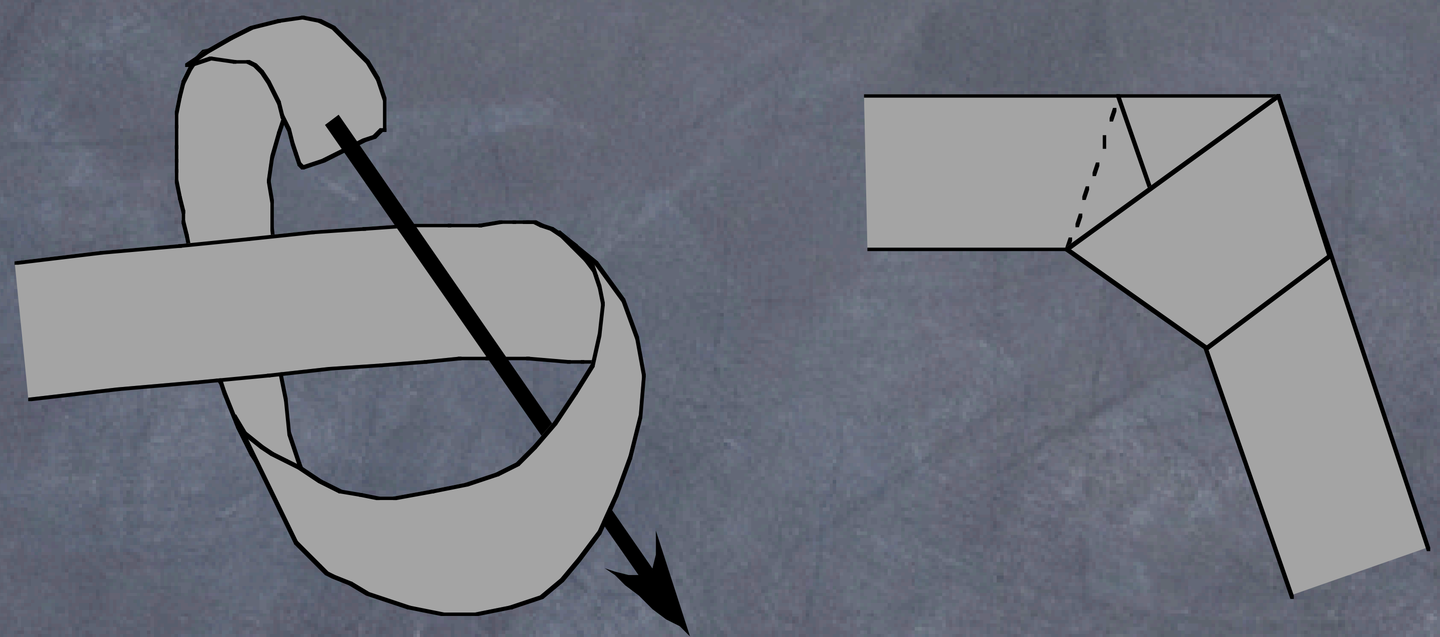
or...

$\alpha \in \mathbb{C}$ is SE&C constructible if and only if $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2^n$ for some $n \geq 0$. In other words, α is algebraic over $\mathbb{Q}$ and the degree if its minimal polynomial over $\mathbb{Q}$ is a power of 2.

Origami version:

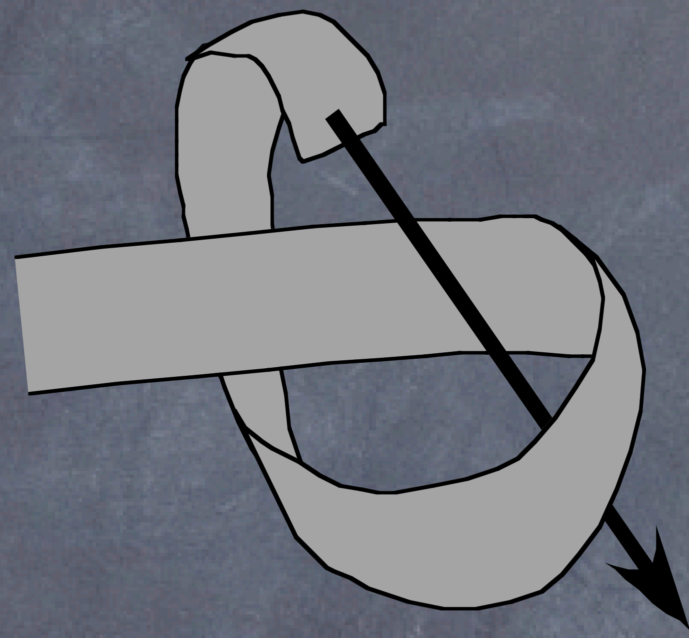
Let $\alpha \in \mathbb{C}$ be algebraic over $\mathbb{Q}$, and let $L \supset \mathbb{Q}$ be the splitting field of the minimal polynomial of α over $\mathbb{Q}$. Then α is origami constructible from our list of BOOs if and only if $[L : \mathbb{Q}] = 2^a 3^b$ for some integers $a, b \geq 0$.

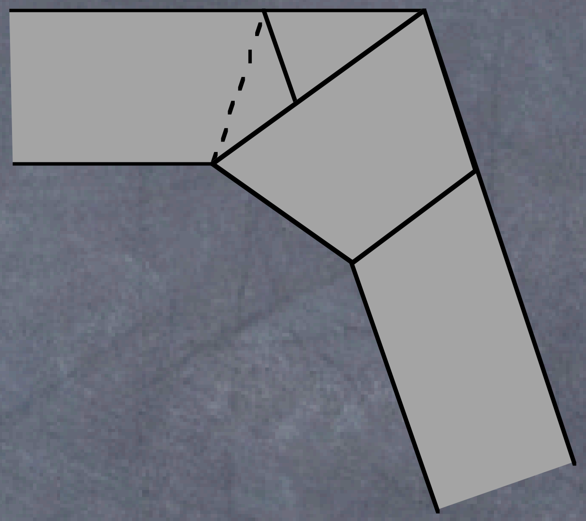
Folding Knots



Take a strip of paper and tie it in a knot.
What do you get?

Folding Knots



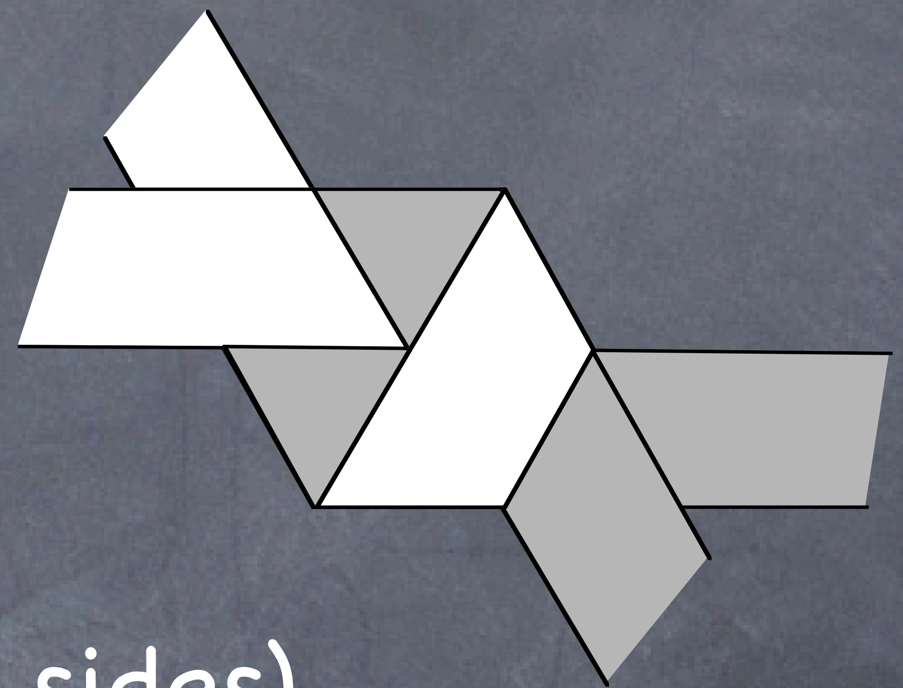


The knot forms a perfect pentagon!

Can you make a hexagon knot?

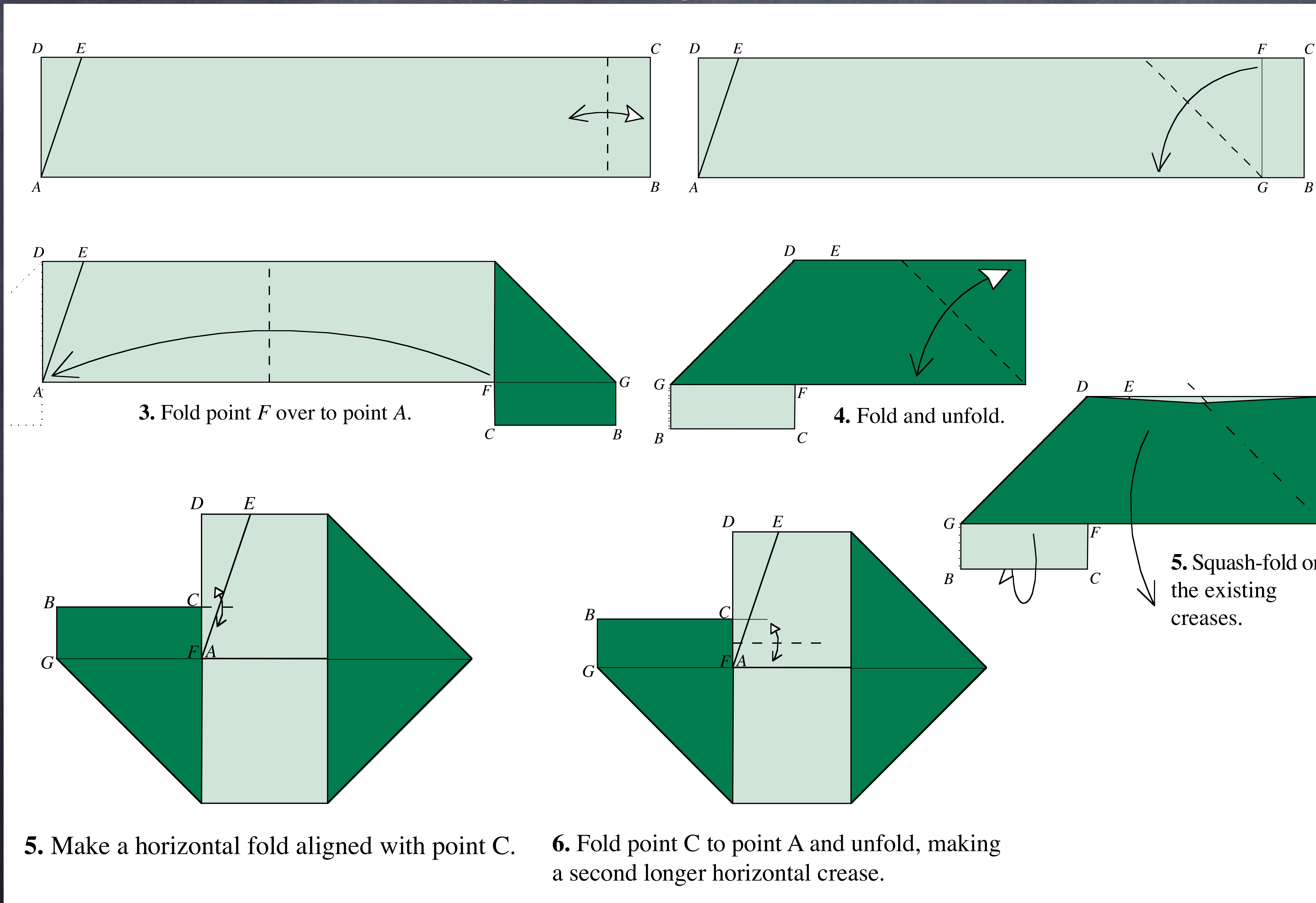
Answer: NO!!!

But you can make a hexagon
from two strips of paper:



And you can make a heptagon (7 sides)
knot from one:

Robert Lang's Angle Quintisection



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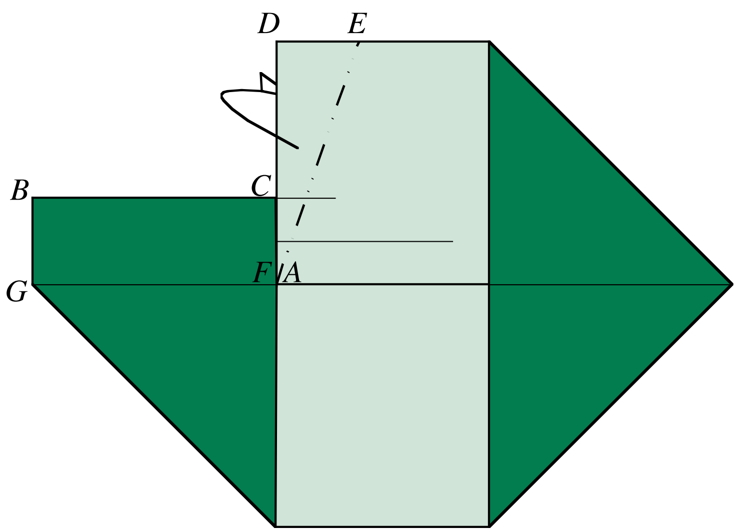
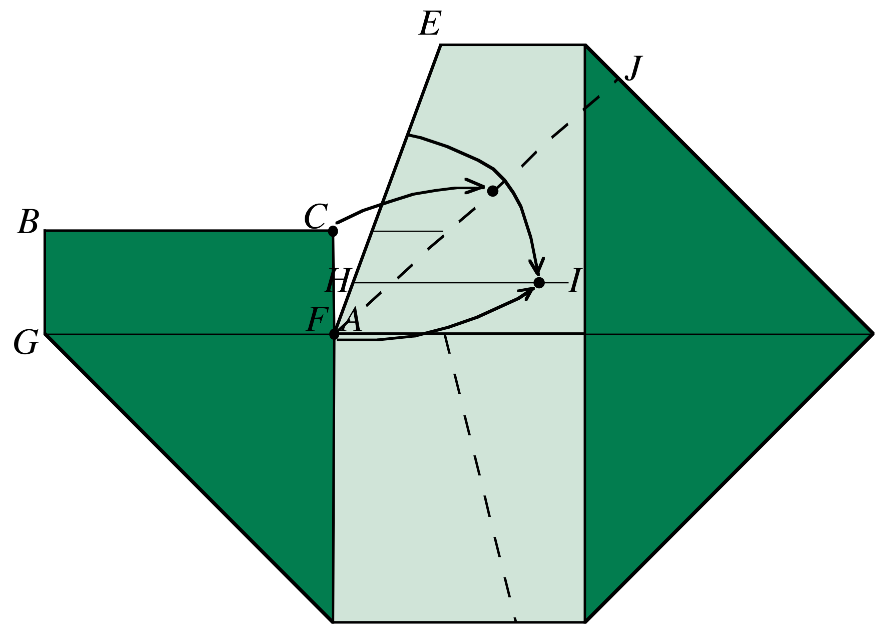
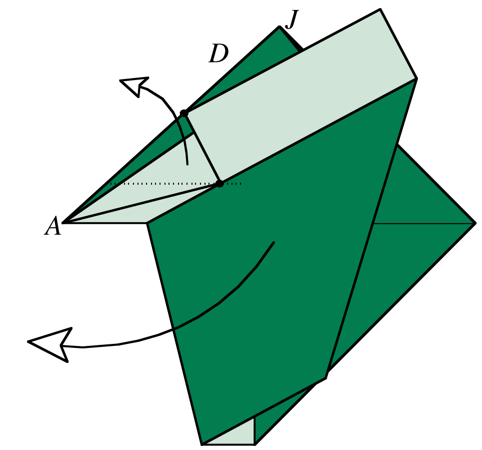


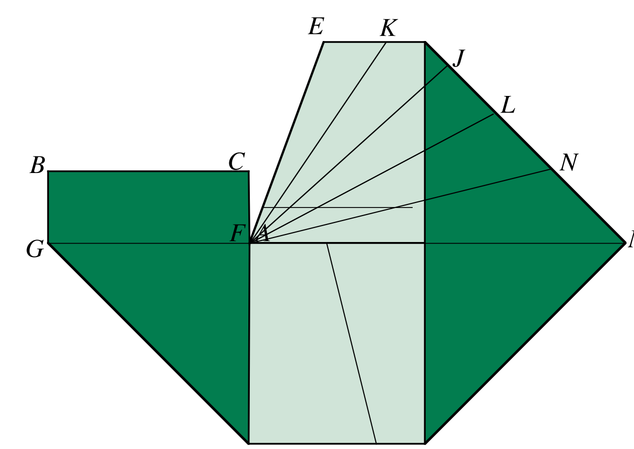
7. Mountain-fold corner D behind.



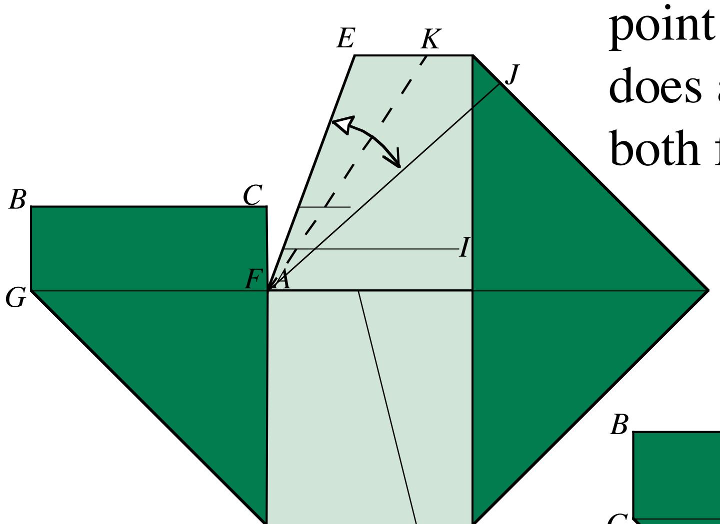


8. Here's what it looks like folded. Yours may not look exactly like this, depending on the angle you used and the length of your strip. Unfold to step 7.

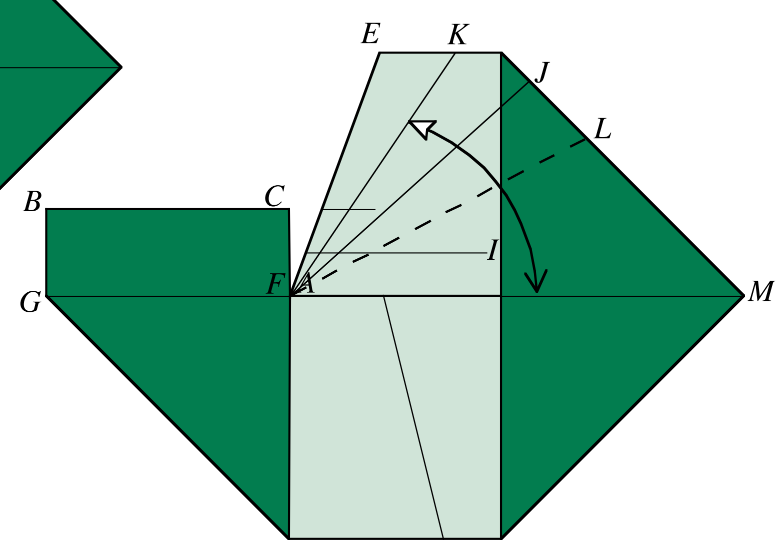
7. Here's where it all happens. Fold edge AE down along crease AJ. At the same time, fold the left flap up so that point F touches crease HI at the same point that edge AE does and point C touches crease AJ. You will have to adjust both folds to make all the alignments happen at once.



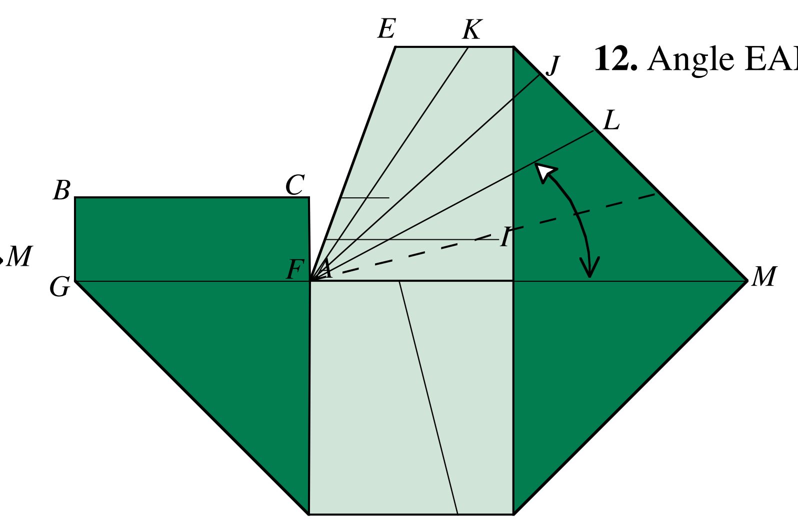
12. Angle EAM is now divided into fifths.



9. Bisect angle EAJ.

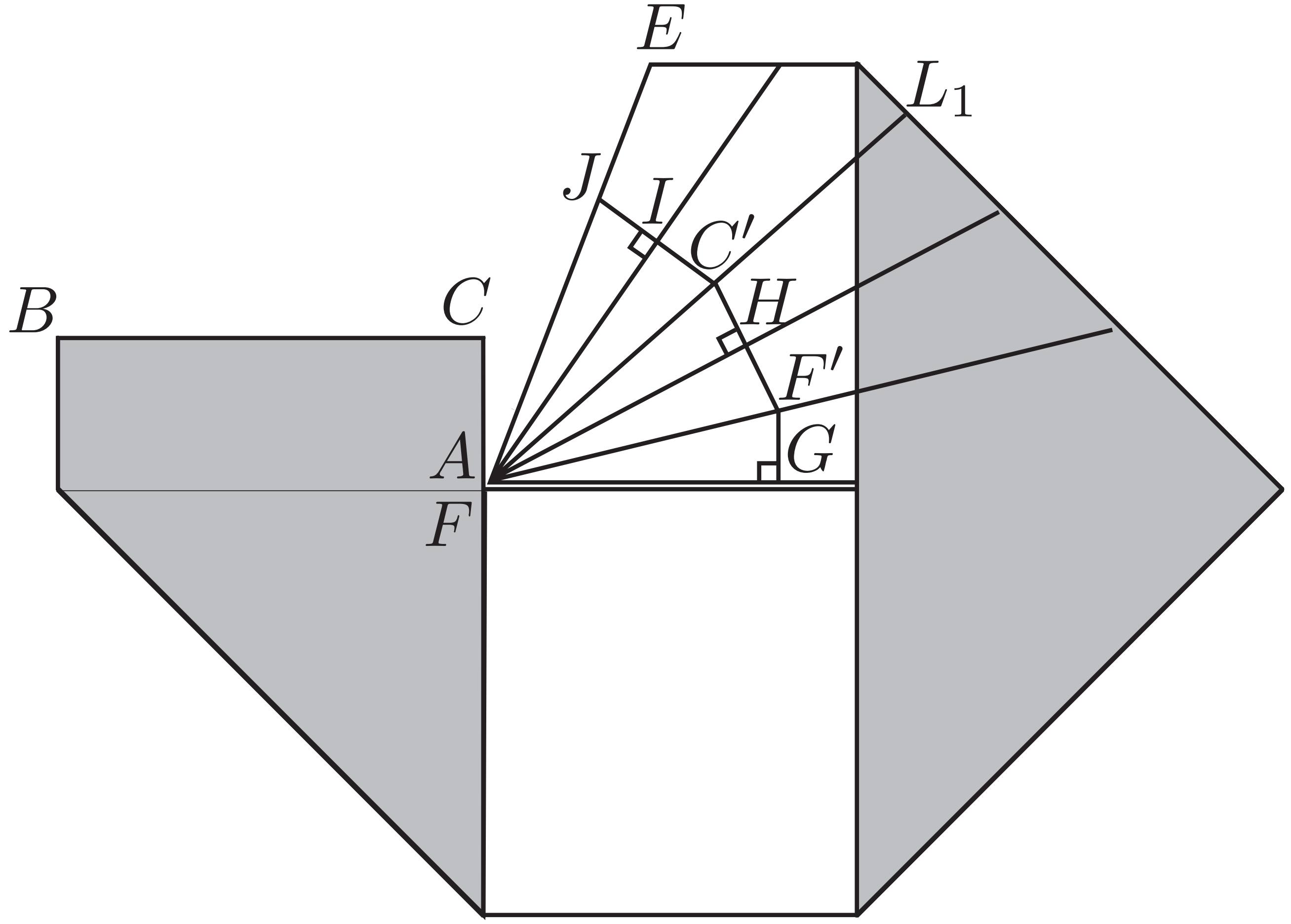


10. Fold crease AK down to AM and unfold.



11. Bisect angle LAM.

Robert Lang's Angle Quintisection

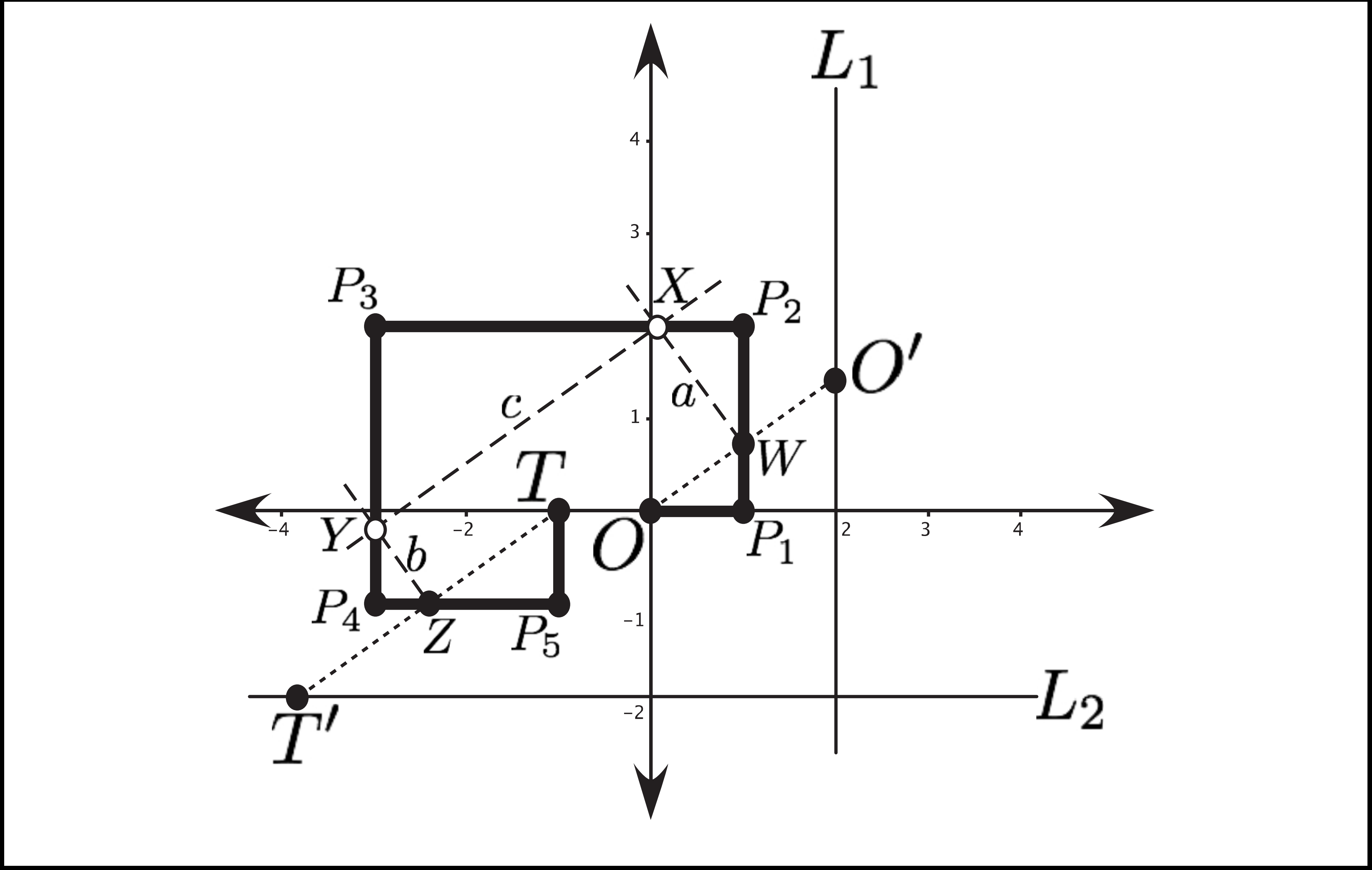


Draw C'F' to be the image of CF under the folding.

Origami Constructions Theorem

How to solve quintics with a 3-fold and Lill's Method:

$$x^5 + 2x^4 + 4x^3 + 3x^2 + 2x + 1 = 0$$



Origami Constructions Theorem

Theorem: (Alperin and Lang, 2009**)**

Every polynomial equation of degree n with real solutions can be solved by an (n-2)-fold.